Two results in T-theory

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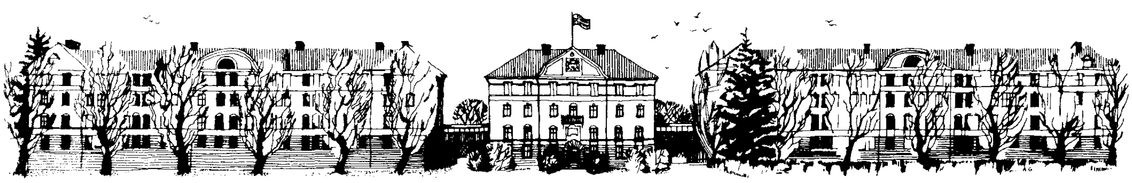

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List of papers

I W. M. Dress, J. Kåhrström, V. Moulton, *A "non-additive" description of $\wp$-adic norms*, submitted in September 2005. Given here with corrections.

II J. Kåhrström, J. Koolen, V. Moulton, *Classification of six-point metrics — revisited*, draft paper.

1 Introduction

This thesis consists of two papers, both dealing with (quite different) aspects of T-*theory*, which can be described as the study of "tight maps". T-theory traces its origin to the late 1970's, where it turned out to answer questions raised in *phylogenetic analysis*, i.e. the study of the evolutionary history of a set of organisms. The main problem being; given a set of species, and a notion of evolutionary distance between any two species within the set, is it possible to accurately represent this information as a phylogenetic tree, with the edge lengths in the tree reflecting the evolutionary distance between the species?

Many ways of providing solutions to this problem has emerged, one of them being T-theory. However, T-theory has also proved itself valuable in a wide range of problems, which the two papers in this thesis are evidence of.

The main tool in T-theory is the T-*construction*, which to any set X and symmetric map $v : X^n \to \mathbb{R}$, $n \in \mathbb{Z}_+ := \{1, 2, 3, \dots\}$, associates a geometric structure called the *tight span* of X with respect to v. The tight span captures a lot of the structure that is endowed on X by the map v. For example, if X is the set of leaves in a tree (in the combinatorial sense of the words), and $v : X^2 \to \mathbb{R}$ is the metric on X induced by the tree, then the tight span of X with respect to d recaptures the underlying tree structure.

The name T-construction stems from the abundance of the letter 'T' as first letter of words like 'tight span', 'tight map' and 'tree'. For an overview of T-theory, see [4].

2 A crash course in T-theory

As mentioned in the introduction, the main tool in T-theory is the T-construction, defined in its full generality as follows. Let X be any non-empty set, and let $v : X^n \to \mathbb{R}$, $n \in \mathbb{Z}_+$, be symmetric, i.e.

$$v(x_1, \dots, x_n) = v(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for all permutations σ of the set $\{1, \dots, n\}$. A map $f : X \to \mathbb{R}$ is *tight* over (X, v) if

$$f(x) = \max_{x_2,\dots,x_n \in X} \left\{ v(x, x_2, \dots, x_n) - \sum_{i=2}^{n} f(x_i) \right\}$$

for all $x \in X$. The *tight span* of (X, v), denoted by $T(X, v)$, is the set of all tight maps over (X, v). However, in this general case, not much can be said about $T(X, v)$. It seems like the interesting cases arise when v satisfies some boundedness condition, the two most studied being when (X, v) is a metric space, and when (X, v) is a valuated matroid.

2.1 The tight span of a metric space

Let (X, d) be a metric space, i.e. X is a non-empty set and $d : X^2 \to \mathbb{R}_{\geq 0}$ satisfying

- $d(x, x) = 0$ for all $x \in X$,
- d is symmetric, and
- $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

For simplicity we assume that X is finite, with $|X| = n$. Then a map $f : X \to \mathbb{R}$ is tight if,

$$f(x) = \max_{y \in X} \big(d(x, y) - f(y)\big),$$

i.e. if

$$f(x) + f(y) \geq d(x, y)$$

for all $x, y \in X$, and for each $x \in X$ there is a $y \in X$ with

$$f(x) + f(y) = d(x, y).$$

We can define a metric d_T on $T(X, d)$ through

$$d_T(f, f') = \max_{x \in X} |f(x) - f'(x)| .$$

Furthermore, the mapping $X \to T(X, d), x \mapsto f_x$, where $f_x(y) = d(x, y)$, is an isomorphism from (X, d) into $\big(T(X, d), d_T\big)$. In particular, for $x \in X$ and $f \in T(d)$ we have

$$d_T(f_x, f) = \max_{y \in X} |f_x(y) - f(y)| = \max_{y \in X} |d(x, y) - f(y)| = f(x).$$

For example, let $X = \{x, y, z, w\}$, and let $d : X^2 \to \mathbb{R}$ be given by $d(x, y) = d(y, z) = d(z, w) = d(x, w) = 3$ and $d(x, z) = d(y, w) = 4$. Then $T(X, d)$ can be shown to be the metric space given in Figure 2.1, where each edge has length 1 and the central square has the "city block"-metric, i.e. geodesics are given by vertical and horizontal line segments.

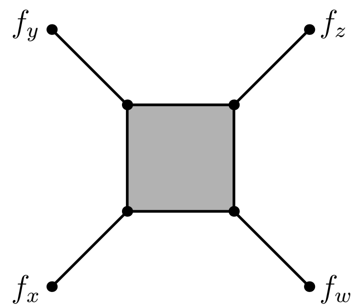

Figure 1: The tight span of the metric described in Section 2.1.

2.2 The tight span of a valuated matroid

Let $(E, v)_n$ be a *valuated matroid of rank* n, with $n \in \{2, 3, \dots\}$, i.e. E is a non-empty set and v is a map $v : E^n \to \mathbb{R} \cup \{-\infty\}$ satisfying

- v is symmetric,
- $v(e_1, \dots, e_n) = -\infty$ for every sequence $e_1, \dots, e_n \in E$ with $|\{e_1, \dots, e_n\}| < n$, and
- for all $e_1, \dots, e_n, f_1, \dots, f_n \in E$, the inequality

$$\begin{aligned} &v(e_1, \dots, e_n) + v(f_1, \dots, f_n) \\ &\qquad \le \max_{1 \le i \le n} \{v(e_1, \dots, e_{i-1}, f_1, e_{i+1}, \dots, e_n) + v(e_i, f_2, \dots, f_n)\} \end{aligned}$$

  holds.

Now, recall that an $\mathbb{R}$-tree (T, d) is a metric space satisfying

- for each $p, q \in T$ there is a unique isometry $\psi_{pq} : [0, d(p, q)] \hookrightarrow T$ with $\psi_{pq}(0) = p$ and $\psi_{pq}\big(d(p, q)\big) = q$,
- all continuous injective maps $[0, 1] \hookrightarrow T$, $t \mapsto p_t$ satisfy $d(p_0, p_1) = d(p_0, p_t) + d(p_t, p_1)$ for all $t \in [0, 1]$, and
- for each $p \in T$ there is an isometric embedding $\psi : \mathbb{R} \to T$ with $\psi(0) = p$.

In [9] it is shown that the tight span of a rank 2 valuated matroid is always an infinite $\mathbb{R}$-tree. In general, the tight span of a rank n valuated matroid is an $(n-1)$-dimensional analogue of an $\mathbb{R}$-tree, satisfying for example that for each point $p \in T(E, v)$ there is an isometry $\psi : \mathbb{R}^{n-1} \hookrightarrow T(E, v)$ with $\psi(0, \ldots, 0) = p$.

3 Paper 1 — A "non-additive" description of $\wp$-adic norms

Let F be a discretely valuated field, i.e. there is map $|\cdot| : F \to \mathbb{Z} \cup \{-\infty\}$ satisfying

- $|\cdot|$ restricted to the non-zero elements of F is a surjective homomorphism from the multiplicative group of F to the group $\mathbb{Z}$ under addition,
- $|0| = -\infty$,
- $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in F$.

Now consider the vector space F^n, $n \in \mathbb{N}$. A *$\wp$-adic norm* is a map $f : F^n \to \mathbb{R} \cup -\infty$ satisfying

- $f(kx) = |k| + f(x)$ for all $k \in F$, $x \in F^n$,
- $f(x + y) \leq \min f(x), f(y)$ for all $x, y \in F^n$, and
- $f(x) = -\infty$ if and only if $x = 0$.

$\wp$-adic norms, defined in this way, has been studied since the late 1940's (see [3, 8]). At first the norms themselves were of interest, and later the whole space of $\wp$-adic norms on F^n (see [5]) were studied. In 1984, F. Bruhat and J. Tits showed in in [1] that the space of $\wp$-adic norms on F^n is a *geometric realization* of the *euclidean building* for the general linear group of F^n. We don't really need to understand what this means, but rather note that in [9], W. Terhalle showed that the tight span of a certain valuated matroid $M(F^n)$ connected to the vector space F^n is also a geometric realization of the same building.

This suggested a connection between this tight span and $\wp$-adic norms. In [6], the author showed that the elements of the tight span of $M(F^n)$ considered by W. Terhalle are in fact $\wp$-adic norms on F^n.

In [10], W. Terhalle considers another set of maps, the *hull* of a valuated matroid, which includes the tight maps. In particular, for a map f in the hull of a valuated matroid, the map $f + c$ is also in the hull for all $c \in \mathbb{R}$, a property that is also true for $\wp$-adic norms, but not for tight maps.

In this paper we give a complete characterization of $\wp$-adic norms in terms of the maps in the hull of $M(F^n)$. In particular, we get a description of $\wp$-adic norms that does not explicitly refer to the vector nature of F^n, hence the name *non-additive*. This paper has been submitted to the journal "Contributions to Algebra and Geometry" in Septembre 2004.

4 Paper 2 — Classification of six point metrics — revisited

This paper deals with the more "concrete" subject of finite metrics. The aim of the paper was, initially, to classify the different types of metrics on six points using techniques presented in [4], concluding the work started in [7] where the *prime* metrics on six points were classified. Here, the main idea is the study of *coherent decompositions*, which is a decomposition of a metric d on a finite set X as

$$d = d_1 + \cdots + d_k,$$

for metrics $d_1, \ldots, d_k$ on X, where the structure of $T(X, d)$ is decomposed into $T(X, d_1), \ldots, T(X, d_k)$.

However, during the course of working on this, B. Sturmfels and J. Yu published a classification of the six point metrics (see [11]). Unfortunately, their paper gave little explanation of *why* the classification (and in particular the definition of *generic* metrics) is a natural one, using advanced theory on triangulations of polytopes. Hence the purpose of this paper is twofold; (i) to give a detailed account for why the classification is a reasonable one, and to present an independent verification of their result.

We have thus completed our classification, and our results agree completely with those of B. Sturmfels and J. Yu.

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A ‘non-additive’ characterization of $\wp$-adic norms

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1 Introduction

1.1 Notations, Definitions, and Standard Facts

In this note, we consider a $\wp$-adic field F, i.e., a field F (with $0 = 0_F$ and $1 = 1_F$ denoting, respectively, the additively and multiplicatively neutral element in F) together with a $\wp$-adic valuation, i.e. a map

$$F \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} : x \mapsto |x| = |x|_\wp$$

that satisfies following conditions for all $x, y \in F$:

(**V0**) $|x| = -\infty$ if and only if $x = 0$,

(**V1**) $|x \cdot y| = |x| + |y|$, and

(**V2**) $|x + y| \leq \max\{|x|, |y|\}$.

Note that $|1| = |-1| = 0$ must hold in view of (**V0**), (**V1**), and $1^2 = (-1)^2 = 1$ which in turn implies that

$$|x| = |-x| \text{ and } |y^{-1}| = -|y|$$

holds for all $x \in F$ and $y \in F^*$ $(:= F \setminus \{0\})$. Thus, given any natural number $n \in \mathbb{N} := \{1, 2, \dots\}$ and any n-tuple $e_1, \dots, e_n$ of row vectors $e_1 = (e_{11}, e_{12}, \dots, e_{1n})$, $e_2 = (e_{21}, e_{22}, \dots, e_{2n})$, $\dots$, $e_n = (e_{n1}, e_{n2}, \dots, e_{nn})$ in F^n, the value

$$\|e_1, e_2, \dots, e_n\| = \|e_1, e_2, \dots, e_n\|_\wp := |\det(e_{ij})_{i,j=1,\dots,n}|$$

of the determinant of the corresponding square $n \times n$-matrix

$$M(e_1, e_2, \dots, e_n) := \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix}$$

only depends on the set $\{e_1, \dots, e_n\}$ of row vectors and not on the given labeling. Thus, we can associate an element $\|B\| = \|B\|_\wp \in \overline{\mathbb{R}}$ to every finite subset B of F^n of cardinality at most n, by putting

$$\|B\| = \|B\|_\wp := \|e_1, e_2, \dots, e_n\|.$$

in case B is of the form $B = \{e_1, e_2, \ldots, e_n\}$ for some elements $e_1, e_2, \ldots, e_n \in E$ and $\|B\| := -\infty$ in any other case.

Note that B is a basis of F^n if and only if $\|B\| \neq -\infty$ holds and that, in particular, $\|B\| = -\infty$ holds for any subset B of F^n as above with $\#B < n$.

Note also that, given any $n \times n$ square matrix $M = (m_{ij})_{i,j=1,\ldots,n}$ with entries from F, we have

$$|\det(M)| \leq \max_{\sigma \in \Sigma_n} \left\{ \sum_{i=1}^{n} |m_{i\sigma(i)}| \right\}$$

where Σ_n denotes the symmetric group of order n, i.e. the group consisting of all permutations σ of the set $[n] := \{1, \ldots, n\}$.

Finally, given any non-empty subset B of F^n, any element $b \in B$, and any element $e \in F^n$, we define the *modified subset* $B \overset{b}{\leftarrow} e$ by

$$B \overset{b}{\leftarrow} e := (B \setminus \{b\}) \cup \{e\},$$

and we denote, in case B is a basis of F^n, the unique map $k : F^n \times B \to F$ for which

$$e = \sum_{b \in B} k(e, b) b$$

holds for every $e \in F^n$ by k_B. Clearly, we have

$$\|B \overset{b}{\leftarrow} e\| = |k_B(e, b)| + \|B\| \tag{1}$$

for every basis B of F^n, every $b \in B$, and every $e \in F^n$, and we have

$$\|B'\| = \|B\| + |\det\big(K(B, B')\big)|$$

for any two bases B, B', where $K(B, B')$ is the $B \times B'$-matrix

$$K(B, B') := \big(k_B(b', b)\big)_{b' \in B', b \in B}$$

and, hence,

$$\|B'\| \leq \|B\| + \max_{\sigma \in \Sigma(B', B)} \left\{ \sum_{b' \in B} |k_B\big(b', \sigma(b')\big)| \right\} \tag{2}$$

where σ runs over the set $\Sigma(B', B)$ of bijections from B' onto B.

1.2 Norms

Recall that, continuing with the notations and definitions from Section 1.1, a *($\wp$-adic) norm* on F^n is a map $p : F^n \to \overline{\mathbb{R}}$ for which

($\mathbf{N}$) $p(xe + yf) \leq \max\{|x| + p(e), |y| + p(f)\}$

holds for all $x, y \in F$ and $e, f \in F^n$.[1]

Note that a map $p : F^n \to \overline{\mathbb{R}}$ is a norm if and only if

($\mathbf{N1}$) $p(xe) \leq |x| + p(e)$, and

($\mathbf{N2}$) $p(e + f) \leq \max\{p(e), p(f)\}$

holds $x \in F$ and $e, f \in F^n$, and that a map $p : F^n \to \overline{\mathbb{R}}$ satisfies the condition ($\mathbf{N1}$) for all $x \in F$ and $e \in F^n$ if and only if

($\mathbf{N1'}$) $p(xe) = |x| + p(e)$

holds for all $x \in F$ and $e \in F^n$, because ($\mathbf{N1}$) and ($\mathbf{N1'}$) are obviously equivalent in case $x = 0$ while, in case $x \neq 0$, we have

$$\begin{aligned} p(xe) &\leq |x| + p(e) \\ &= |x| + p(x^{-1}xe) \\ &\leq |x| + |x^{-1}| + p(xe) \\ &= |x| - |x| + p(xe) \\ &= p(xe) \end{aligned}$$

and, hence, $p(xe) = |x| + p(e)$ in case ($\mathbf{N1}$) holds for all $x \in F$ and $e \in F^n$.

Recall also that maps $p : F^n \to \overline{\mathbb{R}}$ for which

($\mathbf{H}$) $\|B\| + p(e) \leq \max_{b \in B} \left\{ \|B \overset{b}{\leftarrow} e\| + p(b) \right\}$

holds for every $e \in F^n$ and every non-empty subset B of F^n of cardinality at most n (or, equivalently, for every basis B of F^n), were studied in [15]. Thus, it seems worth noting that there is no difference between such maps and norms. Indeed, we will show in the next section that the following holds:

[1] More precisely, such maps have been called "semi-norms" in [10] while norms were defined to be semi-norms for which $p(e) \neq -\infty$ holds for every non-zero vector e in F^n; however, we propose to drop this specific condition in the definition of a ($\wp$-adic) norm to avoid having to talk about 'semi'-objects more often than about objects.

Theorem 1.1. *Given, as above a field F with a $\wp$-adic valuation, a number $n \in \mathbb{N}$, and a map $p : F^n \to \overline{\mathbb{R}}$, the following assertions are equivalent:*

(i) *p is a norm,*

(ii) *the inequality*

  $(\mathbf{H}')$ $p(e) \leq \max_{b \in B} \{|k_B(e,b)| + p(b)\}$

  holds for every basis B of F^n and every element $e \in F$,

(iii) $(\mathbf{H})$ *holds for every $e \in F^n$ and every basis B of F^n.*

Remark 1.2. *Note that neither the condition* $(\mathbf{H})$ *nor the condition* $(\mathbf{H}')$ *refer explicitly to vector addition in F^n and that* $(\mathbf{H})$ *does not even refer to the* coordinate maps $k_B(e,b)$ *(refered to in* $(\mathbf{H}')$*) which explains why we refer to the characterizations of norms provided by Theorem 1.1* (*ii*) *and, in particular,* (*iii*) *as 'non-additive' characterizations of norms*

1.3 Tight Maps

Next, recall that, continuing with the notations introduced above, a map $p : F^n \to \overline{\mathbb{R}}$ is called *tight* if the following conditions hold:

(T1) $\|B\| \leq p(B) := \sum_{b \in B} p(b)$ holds for all non-empty subsets B of F^n of cardinality at most n,

(T2) there exists, for every non-zero element $e \in F^n$, a basis B of F^n containing e with $p(B) = \|B\|$.

Note that a map p is tight if and only if $p(e) \neq -\infty$ holds for every non-zero vector e in F^n and we have

$$(\mathbf{T})\quad p(e) = \max\left\{ \|e, e_2, \ldots, e_n\| - \sum_{i=2}^{n} p(e_i) : e_2, \ldots, e_n \in F^n \setminus \{\underline{0}\} \right\}$$

for every $e \in F^n$.

In [9], it was noted that every tight map satisfies the condition $(\mathbf{H})$ for every $e \in F^n$ and every non-empty subset B of F^n of cardinality at most n. Consequently, Theorem 1.1 implies that any such map must be a norm. Our second result characterizes those norms $p : F^n \to \overline{\mathbb{R}}$ that are tight:

Theorem 1.3. *Continuing with the definitions and notations given in the introduction, a map $p : F^n \to \overline{\mathbb{R}}$ is tight if and only if there exist a basis B and a map $c : B \to \mathbb{R}$ with $\sum_{b\in B} c(b) = \|B\|$ such that*

$$p(e) = \max_{b\in B} \{|k_B(e, b)| + c(b)\}$$

holds for all $e \in F^n$ in which case

$$p(e) = \max_{b\in B} \{|k_B(e, b)| + p(b)\}$$

holds for all $e \in F^n$ and for every basis B with $p(B) = \|B\|$.

Theorem 1.3 implies in particular that, given any basis B of F^n and any map $c : B \to \mathbb{R}$, the associated map

$$p = p_{(B,c)} : F^n \to \overline{\mathbb{R}} : e \mapsto \max_{b\in B} \{|k_B(e, b)| + c(b)\}$$

is always a norm, whether or nor $\sum_{b\in B} c(b) = \|B\|$ holds. Such norms will henceforth be called *proper norms*. Clearly, $p(e) \neq -\infty$ holds for every proper norm and every non-zero vector $e \in F^n$. More specifically, Theorem 1.3 implies the following

Theorem 1.4. *Given any map $p : F^n \to \overline{\mathbb{R}}$, the following assertions are equivalent:*

(i) *p is a proper norm,*

(ii) *there exist a tight map $p_0 : F^n \to \overline{\mathbb{R}}$ and a real number $\gamma \in \mathbb{R}$ with $p(e) = \gamma + p_0(e)$ for all $e \in F^n$,*

(iii) *there exists, for every $e \in F^n \setminus \{\underline{0}\}$, a basis B_e with $p(B_e) \neq -\infty$, $e \in B_e$, and $\|B_e\| - p(B_e) \geq \|B\| - p(B)$ for every basis B.*

Remark 1.5. *It is easy to see that there exist non-proper norms $p : F^n \to \overline{\mathbb{R}}$ with $p(e) \neq -\infty$ for every non-zero vector $e \in F^n$ in case F is not a* complete *$\wp$-adic field with respect to its valuation $|\cdot| : F \to \overline{\mathbb{R}}$. Yet, one can show (cf.[10]) that in case F is a complete $\wp$-adic field relative to a discrete $\wp$-adic valuation (i.e. a map $|\cdot| : F \to \overline{\mathbb{R}}$ with $|x| \in \mathbb{Z}$ for all non-zero elements $x \in F$), then a norm p is proper if and only if $p(e) \neq -\infty$ holds for every non-zero vector $e \in F^n$.*

Theorem 1.1 will be established in the next section, Theorem 1.3 in Section 3, Theorem 1.4 in Section 4, and the relation of these results with tight-span theory and the theory of affine buildings (for the special linear group) will be discussed in the last section.

2 Proof of Theorem 1.1

(i) ⇒ (ii): Let $p : F^n \to \overline{\mathbb{R}}$ be a norm and let $B = \{b_1, \dots, b_n\}$ be a basis of F^n. By successively applying (**N**), we obtain

$$\begin{aligned}
p(e) &= p\left(\sum_{i=1}^{n} k_B(e, b_i) b_i\right) \\
&\leq \max\left\{|k_B(e, b_1)| + p(b_1), p\left(\sum_{i=2}^{n} k_B(e, b_i) b_i\right)\right\} \\
&\leq \max\left\{|k_B(e, b_1)| + p(b_1), |k_B(e, b_2)| + p(b_2), p\left(\sum_{i=3}^{n} k_B(e, b_i) b_i\right)\right\} \\
&\leq \cdots \leq \max_{b \in B} \{|k_B(e, b)| + p(b)\},
\end{aligned}$$

i.e. (**H**$'$) holds for every $e \in F$ and every basis of F^n, as claimed.

(ii) ⇒ (i): Now assume that (ii) holds for a map $p : F^n \to \overline{\mathbb{R}}$ and consider elements $x \in F$ and $e, f \in F^n$. If $e = \underline{0}$, we have $p(xe) = |x| + p(e) = -\infty$. Otherwise, choose a basis B that contains e and note that

$$p(xe) \leq \max_{b \in B} \{|k_B(xe, b)| + p(b)\} = |x| + p(e)$$

holds in view of $k_B(xe, e) = |x|$ and $k_B(xe, b) = 0$ for every $b \in B - \{e\}$.

Similarly, if $e, f \in F^n$ are linearly dependent and, say, $f = ze$ holds for some $z \in F$, we have,

$$\begin{aligned}
p(e + f) &= p\big((1 + z)e\big) = |1 + z| + p(e) \\
&\leq \max\{|1|, |z|\} + p(e) = \max\{|1| + p(e), |z| + p(e)\} \\
&= \max\{p(e), p(ze)\} = \max\{p(e), p(f)\}.
\end{aligned}$$

Otherwise, if e and f are linearly independent, choose a basis B that contains both, e and f, and note that

$$p(e+f) \leq \max_{b \in B} \big(|k_B(e+f,b)| + p(b)\big) = \max\big(p(e), p(f)\big)$$

must hold as $k_B(e+f,b) = 1$ now holds for $b \in \{e,f\}$, and $k_B(e+f,b) = 0$ for every other $b \in B$.

This shows that both, (**N1**) and (**N2**), hold for p implying that this map is a norm.

The equivalence of (ii) and (iii) is obvious in view of (1).

3 Proof of Theorem 1.3

We will first show that, given a basis $B = \{e_1, e_2, \ldots, e_n\}$ of F^n and a map $c : B \to \overline{\mathbb{R}}$ with $\sum_{b \in B} c(b) = \|B\|$, the map $p : F^n \to \overline{\mathbb{R}}$ defined by

$$p(e) = \max_{b \in B} \{|k_B(e,b)| + c(b)\}$$

for all $e \in F^n$ is a tight map: Note first that in view of the fact that

$$k_B(b,b') = \begin{cases} 1 & \text{if } b = b', \\ 0 & \text{if } b \neq b', \end{cases}$$

holds for all $b, b' \in B$, we have $p(b) = c(b)$ for all $b \in B$. In particular, we have $p(B) = \|B\|$. We will show that (**T1**) and (**T2**) both hold for p. To show that (**T1**) holds, let B' be any basis of F^n, and note that (2) implies

$$\begin{aligned}
\|B'\| &\leq \|B\| + \max_{\sigma \in \Sigma(B',B)} \left\{ \sum_{b' \in B'} \big|k_B\big(b', \sigma(b')\big)\big| \right\} \\
&= \max_{\sigma \in \Sigma(B',B)} \left\{ \|B\| + \sum_{b' \in B'} \big|k_B\big(b', \sigma(b')\big)\big| \right\} \\
&= \max_{\sigma \in \Sigma(B',B)} \left\{ \sum_{b' \in B'} \Big(\big|k_B\big(b', \sigma(b')\big)\big| + c(b') \Big) \right\}
\end{aligned}$$

and hence, in view of

$$\max_{\sigma\in\Sigma(B',B)}\left\{\sum_{b'\in B'}\big(|k_B\big(b',\sigma(b')\big)|+c(b)\big)\right\}$$
$$\leq\sum_{b'\in B'}\max_{b\in B}\{|k_B(b',b)|+c(b)\}=\sum_{b'\in B'}p(b')=p(B'),$$

we have $\|B'\|\leq p(B')$, as required.

To show that p also satisfies (**T2**), let e be any non-zero vector in F^n and choose any element $b_0\in B$ with

$$p(e)=|k_B(e,b_0)|+p(b_0).$$

Then $B':=B\xleftarrow{b_0}e$ is a basis that contains e and

$$\|B'\|=\|B\|+|k_B(e,b_0)|=\sum_{b\in B}p(b)+|k_B(e,b_0)|=$$
$$\sum_{b\in B\setminus\{b_0\}}p(b)+p(b_0)+|k_B(e,b_0)|=\sum_{b\in B\setminus\{b_0\}}p(b)+p(e)=p(B') \quad (3)$$

holds, so p satisfies (**T2**).

Conversely, suppose that p is tight. Let B be a basis of F^n with $p(B)=\|B\|$, and let $c:B\to\overline{\mathbb{R}}$ be defined by $c(b)=p(b)$ for all $b\in B$. Then $\sum_{b\in B}c(b)=\|B\|$ holds implying that the map

$$p':F^n\to\overline{\mathbb{R}}:e'\mapsto p'(e):=\max_{b\in B}\{|k_B(e,b)|+c(b)\}$$

is tight. We claim that p and p' must coincide: Indeed, $p'(b)=p(b)=c(b)$ holds for all $b\in B$ and, therefore,

$$p'(e)=\max_{b\in B}\{|k_B(e,b)|+c(b)\}$$
$$=\max_{b\in B}\left\{|k_B(e,b)|+\|B\|-\sum_{\substack{b'\in B\\ b'\neq b}}c(b')\right\}=\max_{b\in B}\left\{\|B\xleftarrow{b}e\|-\sum_{\substack{b'\in B\\ b'\neq b}}p(b')\right\}$$
$$\leq\max_{e_2,\dots,e_n\in F^n\setminus\{\underline{0}\}}\left\{\|e,e_2,\dots,e_n\|-\sum_{i=2}^{n}p(e_i)\right\}=p(e),$$

where the last equality follows from (**T**). Thus $p'(e) \leq p(e)$ for all $e \in F^n$ and, therefore,

$$\begin{aligned} p(e) &= \max_{e_2,\ldots,e_n \in F^n \setminus \{\underline{0}\}} \left\{ \|e, e_2, \ldots, e_n\| - \sum_{i=2}^{n} p(e_i) \right\} \\ &\leq \max_{e_2,\ldots,e_n \in F^n \setminus \{\underline{0}\}} \left\{ \|e, e_2, \ldots, e_n\| - \sum_{i=2}^{n} p'(e_i) \right\} = p'(e). \end{aligned}$$

So, $p = p'$ must hold as claimed. This completes the proof of the theorem.

4 Proof of Theorem 1.4

(i) $\Rightarrow$ (ii): Given any proper norm p, choose a basis B and a map $c : B \to \mathbb{R}$ such that

$$p(e) = \max_{b \in B} \{|k_B(e,b)| + c(b)\}$$

holds for all $e \in E$, put $\gamma := \frac{1}{n}(\sum_{b \in B} c(b) - \|B\|)$, define the map

$$c_0 : B \to \mathbb{R}$$

by $c_0(b) := c(b) - \gamma$ so that

$$\sum_{b \in B} c_0(b) = \sum_{b \in B} c(b) - n\,\gamma = \sum_{b \in B} c(b) - \left(\sum_{b \in B} c(b) - \|B\| \right) = \|B\|$$

holds, and note that $p(e) = p_0(e) + \gamma$ holds for all $e \in E$ for the necessarily tight map

$$p_0 : F^n \to \overline{\mathbb{R}} : e \mapsto p_0(e) := \max_{b \in B} \{|k_B(e,b)| + c_0(b)\}.$$

(ii) $\Rightarrow$ (i): Conversely, suppose that $p(e) = p_0(e) + \gamma$ holds for all $e \in E$ for a tight map p_0 and some $\gamma \in \mathbb{R}$, choose a basis B with $\|B\| = p_0(B)$ so that

$$p_0(e) = \max_{b \in B} \{|k_B(e,b)| + p_0(b)\}$$

holds for all $e \in F^n$, and consider the map $c : B \to \mathbb{R} : b \mapsto p_0(b) + \gamma$. Then,

$$\begin{aligned} p(e) = p_0(e) + \gamma &= \max_{b \in B} \{|k_B(e,b)| + p_0(b)\} + \gamma \\ &= \max_{b \in B} \{|k_B(e,b)| + c(b)\} = p_{(B,c)}(e) \end{aligned}$$

holds for all $e \in F^n$, as required.

(ii) $\Rightarrow$ (iii): Now suppose again that (ii) holds. As p_0 is tight, we have $p_0(B) \neq -\infty$ and $\|B\| \leq p_0(B)$ for all bases B of F^n, and there is a basis B_e for each $e \in F^n \setminus \{\underline{0}\}$ containing e with $\|B_e\| = p_0(B_e)$. Hence,

$$\begin{aligned}
&\max\{\|B\| - p(B) : B \text{ is a basis of } F^n\}\\
&\quad = \max\{\|B\| - p_0(B) : B \text{ is a basis of } F^n\} - n\gamma\\
&\quad \leq -n\gamma = \|B_e\| - p_0(B_e) - n\gamma = \|B_e\| - p(B_e)
\end{aligned}$$

holds as required.

(iii) $\Rightarrow$ (ii): Conversely, suppose that (iii) holds, choose a basis B_e for every $e \in F^n \setminus \{\underline{0}\}$ with $e \in B_e$, $p(B_e) \neq -\infty$ and $\|B_e\| - p(B_e) \geq \|B\| - p(B)$ for every basis B, put

$$\gamma := \frac{1}{n} \max\{p(B') - \|B'\| : B' \text{ is a basis of } F^n\}$$

note that $\gamma = \frac{1}{n}\big(\|B_e\| - p(B_e)\big)$ holds for every $e \in F^n \setminus \{\underline{0}\}$, and consider the map $p_0 : F^n \to \overline{\mathbb{R}} : e \to p(e) - \gamma$. By definition, $p(e) = p_0(e) + \gamma$ holds for all $e \in F^n$. So, it remains to show that p_0 is tight, e.g., that it satisfies (**T1**) and (**T2**). However, we have

$$\begin{aligned}
p_0(B) &= p(B) - \max\{p(B') - \|B'\| : B' \text{ is a basis of } F^n\}\\
&\geq p(B) - \big(p(B) - \|B\|\big)\\
&= \|B\|,
\end{aligned}$$

so p_0 satisfies (**T1**). To see that p_0 satisfies (**T2**), let $e \in F^n \setminus \{\underline{0}\}$ and note that, again essentially by definition, we have

$$p_0(B_e) = p(B_e) - \big(p(B_e) - \|B_e\|\big) = \|B_e\|,$$

so p_0 satisfies also (**T2**). This completes the proof.

5 Discussion: The Relation between Valuated Matroids, Tight Maps, and Buildings

Norms on vector spaces over $\wp$-adic fields have been studied for well over 50 years. At first, interest was focused on the topology that a $\wp$-adic norm induces on the vector space (cf. [3, 11]), but it soon broadened to studying the space of norms (cf. [10]). In [2], F. Bruhat and J. Tits showed in 1984 that, for a discretely valuated $\wp$-adic field F, the space of proper $\wp$-adic norms on the vector space F^n is a 'concrete example', i.e. a geometric realization, of the affine building for the special linear group $\mathrm{SL}_n(F)$.

Curiously enough, an apparently rather distinct geometric realization of this building was described independently in 1992 the details of which we now recall. Here, the starting point was the concept of a *valuated matroid* that surfaced first in 1986 in [6] (as a particularly striking example of a *matroid with coefficients*) and was later addressed explicitly in [1, 7]. According to [6], a valuated matroid of rank $n \geq 2$ is a pair $M = M_n = (E, v)_n$ consisting of a non-empty set E and a non-constant map $v : E \to \overline{\mathbb{R}}$ satisfying the following conditions:

(VM1) For all $e_1, \dots, e_n \in E$ and every permutation σ of $\{1, \dots, n\}$ one has

$$v(e_1, \dots, v_n) = v(e_{\sigma(1)}, \dots, e_{\sigma(n)}),$$

that is, v is totally symmetric.

(VM2) $v(e_1, \dots, e_n) = -\infty$ holds for every sequence of elements $e_1, \dots, e_n \in E$ with $\#\{e_1, \dots, e_n\} < n$.

(VM3) For all $e_1, \dots, e_n, f_1, \dots, f_n \in E$,

$$\begin{aligned} &v(e_1, \dots, e_n) + v(f_1, \dots, f_n) \\ &\qquad \leq \max_{1 \leq i \leq n} \{v(e_1, \dots, e_{i-1}, f_1, e_{i+1}, \dots, e_n) + v(e_i, f_2, \dots, f_n)\} \end{aligned}$$

holds.

As above, this implies that $v(e_1, \dots, e_n) = v(f_1, \dots, f_n)$ holds for any two families $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of elements from E with $\{e_1, \dots, e_n\} = \{f_1, \dots, f_n\}$ and that, in consequence, one can define $v(B)$ for any finite subset B of E of cardinality by putting $v(B) := v(e_1, \dots, e_n)$ in case there are elements $e_1, \dots, e_n \in E$ with $B = \{e_1, \dots, e_n\}$, and $v(B) := -\infty$ else, and one can define B to be a *basis* of M_n if $v(B) \neq -\infty$ holds in which case $\#B = n$ necessarily holds. Further, an element $e \in E$ is called a *loop*[2] if $v(e, e_2, \dots, e_n) = -\infty$ holds for all $e_2, \dots, e_n \in E$.

[2]To avoid having to deal with loops, it is common to add the following requirement:

(VM0) For every $e \in E$, there are elements $e_2, \dots, e_n \in E$ with $v(e, e_2, \dots, e_n) \neq -\infty$.

The celebrated Grassmann-Plücker identity[3]

$$\begin{aligned}&\det(e_1,\ldots,e_n)\cdot\det(f_1,\ldots,f_n)\\=&\sum_{i=1}^{n}\det(e_1,\ldots,e_{i-1},f_1,e_{i+1},\ldots,e_n)\cdot\det(e_i,f_2,\ldots,f_n)\end{aligned}$$

that holds for any $2n$ elements $e_1,\ldots,e_n,f_1,\ldots,f_n\in F^n$, implies that

$$M_n(F):=(F^n,\|.\|_{\wp})_n$$

is a valuated matroid of rank n for any $\wp$-adic field F as considered above whose set of bases is exactly the set of bases of F^n while the origin of F^n is its only loop.

Moreover, it is one of the most basic results of algebraic geometry dating back to the 19th century that a map $D:X^n\to F$ from the n-th cartesian power X^n of a finite set X of cardinality N into F satisfies the identity

$$\begin{aligned}&D(e_1,\ldots,e_n)\cdot D(f_1,\ldots,f_n)\\=&\sum_{i=1}^{n}D(e_1,\ldots,e_{i-1},f_1,e_{i+1},\ldots,e_n)\cdot D(e_i,f_2,\ldots,f_n).\end{aligned}$$

for all $e_1,\ldots,e_n,f_1,\ldots,f_n\in X$ if and only if there exists a map $\varphi=\varphi_D: X\to F^n$ with

$$D(e_1,\ldots,e_n)=\det(\varphi(e_1),\ldots,\varphi(e_n))$$

for all $e_1,\ldots,e_n\in X$ in which case φ is uniquely determined by D up to an F-linear automorphism of F^n of determinant 1 provided D does not vanish identically (i.e., the induced F-linear map $F[\varphi]:F[X]\to F^n$ is surjective). In consequence, the sub-manifold of the projective space $\mathbb{P}(F^{X^n})$ of one-dimensional subspaces $F\cdot D:=\{x\cdot D: x\in F\}$ of F^{X^n} (D a non-vanishing map from X^n into F) that consists of all these one-dimensional subspaces

[3] Sketch of proof: Writing $e_1,\ldots,\hat{e}_i,\ldots,e_n$ for $e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n$ and exchanging f_1 and e_i in case $\det(e_1,\ldots,e_n)=0\neq\det(f_1,e_1,\ldots,\hat{e}_i,\ldots,e_n)$ for some $i\in\{1,\ldots,n\}$, one may assume $\det(e_1,\ldots,e_n)\neq 0$ and, hence, $f_1=\sum_{i=1}^n x_ie_i$ for some $x_1,\ldots,x_n\in F$ which in turn implies $\det(e_1,\ldots,e_n)\det(f_1,\ldots,f_n)+\sum_{i=1}^n(-1)^i\det(f_1,e_1,\ldots,\hat{e}_i,\ldots,e_n)\det(e_i,f_2,\ldots,f_n)=\det(e_1,\ldots,e_n)\det(f_1,\ldots,f_n)+\sum_{i=1}^n(-1)^ix_i\det(e_i,e_1,\ldots,\hat{e}_i,\ldots,e_n)\det(e_i,f_2,\ldots,f_n)=\det(e_1,\ldots,e_n)\big(\det(f_1,\ldots,f_n)-\sum_{i=1}^n x_i\det(e_i,f_2,\ldots,f_n)\big)=0.$

$F \cdot D$ for which D satisfies these identities (i.e., the subspace of $\mathbb{P}(F^{X^n})$ defined by these identities viewed as (homogeneous) polynomials in the 'indeterminate' $D(e_1, \ldots, e_n)$ $(e_1, \ldots, e_n \in X)$) is canonically isomorphic to the Grassmann-Plücker variety $\mathbb{G}_F(N, n)$ of n-dimensional subspaces of the N-dimensional F-vectorspace F^X (identifying each such one-dimensional subspace $F \cdot D$ with the n-dimensional dual of the $(N-n)$-dimensional kernel of φ_D considered as a subspace of F^X using the canonical identification of F^X with its dual). Thus, valuated-matroid theory may also be viewed (as recognized only lately by Bernd Sturmfels et al, cf. [12, 13]) as *the tropical geometry of Grassmann-Plücker varieties over valuated fields.*

Generalizing the definition of tight spans given in [4] in case of metric spaces, one may define the *tight span* $T_{(E,v)}$ of any pair (E, v) consisting of a set E and a map v from the set $\mathbb{P}_{fin}(E)$ of finite subsets of E into $\overline{\mathbb{R}}$ as the set consisting of all maps $p : E \to \overline{\mathbb{R}}$ with

$$p(e) = \sup \left\{ v(B) - \sum_{f \in B \setminus \{e\}} p(f) \;\middle|\; e \in B, v(B) \neq -\infty \right\}.$$

In particular, the tight span of a valuated matroid $M = (E, v)_n$ denoted by T_M, consists of the set of its *tight maps*, i.e., the set of maps $p : E \to \overline{\mathbb{R}}$ satisfying

(**T0′**) $p(e) = -\infty$ for all loops $e \in E$,

(**T1′**) $v(B) \leq p(B) := \sum_{b \in B} p(b)$ holds for all finite subsets B of E,

(**T2′**) there exists, for every element $e \in E$ that is not a loop, a basis B of E containing e with $p(B) = v(B)$.

In [14], it was shown by W. Terhalle that the tight span of the valuated matroid $M_n(F)$ is also a geometric realization of the affine building of the group $\mathrm{SL}_n(F)$ (see also [9]) for any $\wp$-adic field F. This sugested that there must be some connection between $\wp$-adic norms and the maps in the tight span of $M_n(F)$ though, when we began to explore this suggestion, we didn't expect that connection to be that close as it finally turned out and is stated in Theorem 1.4. It follows in particular that, using Theorem 1.4, one can either derive the description of the affine building of the group $\mathrm{SL}_n(F)$ in terms of norms as given by F. Bruhat and J. Tits from that given by W. Terhalle in terms of tight maps or, conversely, the description given by W. Terhalle from that given by F. Bruhat and J. Tits.

As a special case, it follows also, as noted by W. Terhalle, that the tight span of the valuated matroid $M_n(F) := (F^n, v_{\text{triv}})_n$, where v_{triv} is the trivial valuation given by

$$v_{\text{triv}}(B) = \begin{cases} 0 & \text{if } B \text{ is a basis of } F^n, \\ -\infty & \text{otherwize}, \end{cases}$$

is a geometric realization of the spherical building for the group $\mathrm{GL}_n(F)$.

We believe that affine buildings can also be recovered using appropriate "tight-span" constructions for other linear algebraic groups.

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Classification of six-point metrics — Revisited

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Draft version.

1 Introduction

The metric cone $M(X)$ on a finite non-empty set X is a well studied object, see for example [1], [2], and [3]. In [2], a special stratification of $M(X)$ into subcones is suggested using the concept of *coherent decompositions*. A coherent decomposition of a metric d is a decomposition into (possibly simpler) metrics $d_1, \ldots, d_k$ with a certain structure 'compatible' with d, i.e. d_i can be seen as a (sub)structure of d for all $1 \leq i \leq k$. The structure in question is the *tight span* of the metric.

In [8] this stratification of $M(X)$ is explored, with the focus on determining, for metrics d and d' on X, whether d is a coherent component of d. A satisfactory description of the subcones of this stratification is however not presented.

In [11], B. Sturmfels and J. Yu give a complete description of this stratification in the case when $|X| = 6$. They use the fact that this is the same as the *secondary fan of the second hypersimplex*

$$\Delta(n, 2) := \operatorname{conv}\{e_i + e_j \mid 1 \leq i < j \leq n\} \subset \mathbb{R}^n,$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$, an object which is studied in [10].

This paper can be seen as a link between [8] and [11], since we refine the ideas from [8] to give a complete description of the above stratification of the metric cone in terms of coherent decompositions. For completeness, most known results are given with proof. As a conclusion, we use this to give an independent verification of the results on metrics on 6 points by B. Sturmfels and J. Yu.

2 Theory

2.1 The metric cone

For the rest of this paper, we let X be a fixed non-empty finite set. Recall that a *(semi)-metric* on X is a map $d : X^2 \to \mathbb{R}_{\geq 0}$ satisfying, for all $x, y, z \in X$,

(**M1**) $d(x, x) = 0$ for all $x \in X$,

(**M2**) $d(x, y) = d(y, x)$,

(**M3**) $d(x, y) + d(y, z) \geq d(x, z)$.

We denote by $M(X)$ the set of all metrics on X. It is a subset of $\mathbb{R}^{X^2}$, but by (**M1**) and (**M2**), $M(X)$ is contained in an $\binom{|X|}{2}$-dimensional subspace of $\mathbb{R}^{X^2}$, and by (**M3**), $M(X)$ is actually a cone. For this reason, $M(X)$ is called the *metric cone* on X. Note also that if $d_1, \ldots, d_k \in M(X)$, then $d = d_1 + \cdots + d_k \in M(X)$,$k \in \mathbb{N}$, where

$$d(x, y) = (d_1 + \cdots + d_k)(x, y) := d_1(x, y) + \cdots + d_k(x, y)$$

for all $x, y \in X$, called a *decomposition* of d into the metrics $d_1, \ldots, d_k$.

2.2 The associated polyhedral and the tight span

For a metric $d \in M(X)$, we define its *associated polyhedral*, $P(d)$, as the subset of $\mathbb{R}^X$ given by

$$P(d) := \{f : X \to \mathbb{R} \mid f(x) + f(y) \geq d(x, y), \text{ for all } x, y \in X\}.$$

One of the main reasons for studying $P(d)$ is because it contains the subset $T(d) \subset P(d)$ consisting of those elements $f \in P(d)$ satisfying that for each $x \in X$ there is a $y \in Y$ with $f(x) + f(y) = d(x, y)$. $T(d)$ can equivalently be defined as the set of maps $f : X \to \mathbb{R}$ with

$$f(x) = \max_{y \in X} \{d(x, y) - f(y)\}.$$

$T(d)$ is interesting, since we can endow it with a metric $d_{T(d)} : T(d) \to \mathbb{R}$ through

$$d_{T(d)}(f, f') = \max_{x \in X} |f(x) - f'(x)|.$$

For each $x \in X$, we have that the map $f_x : X \to \mathbb{R}$ defined by $f_x(y) = d(x, y)$ for all $y \in X$ is an element of $T(d)$, since by the triangle inequality we have

$$f_x(y) + f_x(y') = d(x, y) + d(x, y') \geq d(y, y'),$$

and for each $y \in X$ we have $f_x(x) + f_x(y) = d(x, x) + d(x, y) = d(x, y)$. Now the map $\phi_d : X \to T(d), x \mapsto f_x$ is an isometry from (X, d) into $T\big(X, d_{T(d)}\big)$. To see this, note that by the triangle inequality we have

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

for all $x, y, z \in X$, and equality is attained if $z = x$. Hence $d(x, y) = \max_{z\in X} |d(x, z) - d(y, z)|$ for all $x, y \in X$,so

$$d_{T(d)}(f_x, f_y) = \max_{z\in X} |f_x(z) - f_y(z)| = \max_{z\in X} |d(x, z) - d(y, z)| = d(x, y).$$

The metric space $\big(T(d), d_{T(d)}\big)$ is called the *tight span* of (X, d).

For $f \in P(d)$, we denote by $[f]$ the minimal face of $P(d)$ containing f, and we let $\mathcal{F}(d)$ denote the poset of faces of $P(d)$, ordered by inclusion. The tight span $T(d)$ can be shown to consist of precisely the compact faces of $P(d)$. Hence, $T(d)$ has a natural decomposition into faces.

Our goal is to classify the metrics on X in terms of how they 'look', and hence we say that two metrics $d, d' \in M(X)$ are of the same *type* if there exists a poset isomorphism $\psi : \mathcal{F}(d) \to \mathcal{F}(d')$ with $\psi([\phi_d(x)]) = [\phi_{d'}(x)]$ for all $x \in X$. We write $d \sim d'$ if d and d' are of the same type. It is immediate that $\sim$ is an equivalence relation. The main part of this paper can be seen as the study of the equivalence classes of this relation, and how the equivalence classes relate to each other.

2.3 The tight equality graph and some of its properties

We start with some basic definitions and notations. Recall that a *graph* is a pair $G = (V, E)$, where V is a non-empty set whose elements are called the *vertices* of G and $E \subseteq V^2$ are the *edges* of G. A sequence $x_1, \dots, x_n \in V$, $n \in \mathbb{N}$, is called a *path* from x_1 to x_n of *length* $(n - 1)$ in G, if $\{x_i, x_{i+1}\} \in E$ for all $1 \leq i < n$. We regard the one-sequence x as a path of length 0 from x to x for all $x \in X$ (note that if $\{x, x\} \in E$ for some $x \in V$, then x, x is a path of length 1 from x to x). A *cycle* in G is a path starting and ending in the same vertex. For each $x \in V$, the *connected component of G containing x* is defined as the set

$$C_x := \{y \in V \mid \text{ there is a path from } x \text{ to } y \text{ in } G\}.$$

A connected component of G is *bipartite* if it contains no cycles of odd length. Finally, recall that for a convex subset $P \subset \mathbb{R}^n$, $n \in \mathbb{N}$, the *relative interior* of P is defined as interior of P relative the affine hull of P.

Let $d \in M(X)$. To help study the faces of $P(d)$ we make the following construction. For each $f \in P(d)$ we associate the set

$$K(f) = \{\{x, y\} \mid x, y \in X \text{ with } f(x) + f(y) = d(x, y)\},$$

and the graph $\Gamma(f) = (X, K(f))$, called the *tight equality graph* of f. Note that this graph allows for loops, i.e. $\{x, x\}$ can be an edge of the graph.

Remark 2.1. For $f \in P(d)$ we have $f \in T(d)$ if and only if $\Gamma(f)$ is not *singular*, i.e. has no connected components on the form $\{x\}$ for some $x \in X$.

Remark 2.2. Let $f \in P(d)$ with $\{x\}$ a component of $\Gamma(f)$ for some $x \in X$. Then if $\{x, x\} \in K(f)$, it follows that $K(f) = \{\{x, x\}\}$. To see this, note that if $\{x\}$ is a component of $\Gamma(f)$, then $\{x, y\} \notin K(f)$ for all $y \in X$, and if $\{x, x\} \in K(f)$ then $f(x) = 0$. Hence for $y, y' \in X$ with $y \neq x$ we get

$$\begin{aligned} d(y, y') &\leq d(y, x) + d(x, y') \\ &\qquad < f(y) + f(x) + f(x) + f(y') = f(y) + f(y)', \end{aligned}$$

i.e. $\{y, y'\} \notin K(f)$. In particular, for such $f \in P(d)$, $\Gamma(f)$ has $|X| - 1$ bipartite components.

The tight equality graph relates to the faces of $P(d)$ through the following lemma and corollary.

Lemma 2.3. *Let $d = d_1 + \cdots + d_k$ be a decomposition of the metric $d \in M(X)$ into the metrics $d_1, \ldots, d_k \in M(X)$, let $f_1 \in P(d_1)$, ..., $f_k \in P(d_k)$ and let $f = f_1 + \cdots + f_k \in P(d)$. Then*

$$K(f) = \bigcap_{i=1}^{k} K(f_i). \tag{1}$$

Proof. We have, for all $x, y \in X$,

$$
\begin{aligned}
f(x) + f(y) &= f_1(x) + \cdots + f_k(x) + f_1(y) + \cdots + f_k(y) \\
&\geq d_1(x,y) + \cdots + d_k(x,y) = d(x,y),
\end{aligned}
$$

so $f \in P(d)$, and since by definition we have $f_i(x) + f_i(y) \geq d_i(x,y)$ for all $1 \leq i \leq k$, with equality if and only if $\{x,y\} \in K(f_i)$, we see that $\{x,y\} \in K(f)$ if and only if $\{x,y\} \in K(f_i)$ for all $1 \leq i \leq k$, and thus (1) holds. $\square$

Corollary 2.4. *Let $f_1, \ldots, f_k \in P(d)$, and let $f \in P(d)$ be in the relative interior of the convex hull of $f_1, \ldots, f_k$. Then*

$$
K(f) = \bigcap_{i=1}^{k} K(f_i).
$$

Proof. Since f is in the relative interior of the convex hull of f_1, $\ldots$, f_k, there are elements $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_+$ with $\alpha_1 + \cdots + \alpha_k = 1$ and

$$
f = \alpha_1 f_1 + \cdots + \alpha_k f_k.
$$

Hence the result follows by applying Lemma 2.3 to the decomposition $d = \alpha_1 d + \cdots + \alpha_k d$. $\square$

Corollary 2.5. *Let $F \subset T(d)$. Then there is a face of $T(d)$ containing F (and in particular, containing the convex hull of F) if and only if the graph (X, K) is non-singular, where*

$$
K := \bigcap_{f \in F} K(f).
$$

From Corollary 2.4 it follows that for two elements $f, f' \in P(d)$ we have $[f'] \subseteq [f]$ if and only if $K(f') \supseteq K(f)$, so $[f'] = [f]$ if and only if $K(f') = K(f)$, and for all $f \in P(d)$ we have

$$
[f] = \{f' \in P(d) \mid K(f) \subseteq K(f')\}.
$$

Thus the set of tight equality graphs of d completely determines the face lattice of $P(d)$. In particular, we have the following result.

Proposition 2.6 (Proposition 4.9, [8])**.** *For two elements $d, d' \in M(X)$ we have $d \sim d'$ if and only if*

$$\{K(f) \mid f \in T(d)\} = \{K(f') \mid f' \in T(d')\}.$$

Proof. The 'if' part follows immediately, since the map $\psi : \mathcal{F}(d) \to \mathcal{F}(d')$, $[f] \mapsto [f']$ where $f' \in T(d)$ with $K(f') = K(f)$ is a poset isomorphism.

For the 'only if' part, let ψ be the poset isomorphism from $\mathcal{F}(d)$ to $\mathcal{F}(d')$, let $x, y \in X$, and let F_{xy} be the minimal face of $P(d)$ containing $\phi_d(x)$ and $\phi_d(y)$. Then $\psi(F_{xy})$ is the minimal face of $P(d')$ containing $\phi_{d'}(x)$ and $\phi_{d'}(y)$, since $\psi\big(\phi_d(x)\big) = [\phi_{d'}(x)]$ and $\psi\big(\phi_d(y)\big) = [\phi_{d'}(y)]$.

The faces on the form $[f]$, $f \in T(d)$ of $T(d)$ with $\{x, y\} \in K(f)$ are precisely the compact faces of F_{xy}. These are mapped by ψ to the compact faces of $\psi(F_{xy})$, which in turn are precisely the faces on the form $[f']$, $f' \in T(d')$ of $T(d')$ with $\{x, y\} \in K(f')$. Thus $\{x, y\} \in K(f)$ if and only if $\{x, y\} \in K(f')$ for elements $f \in T(d)$ and $f' \in T(d')$ with $\psi([f]) = [f']$. $\square$

Corollary 2.7 (Theorem 4.13 and Corollary 4.14, [8])**.** *For a finite set X there exists only finitely many types of metrics on X.*

Proof. For two metrics d and d' on X we have $d \sim d'$ if and only if

$$\{K(f) \mid f \in T(d)\} = \{K(f') \mid f' \in T(d')\},$$

so the result follows from the fact that there are only finitely many graphs on $|X|$ points. $\square$

For an element $f \in P(d)$, the graph $\Gamma(f)$ also holds information about the local structure of $P(d)$ around $[f]$. For example, we have the following useful lemma.

Lemma 2.8 (Lemma 2, [5])**.** *For any $f \in P(d)$, the dimension of $[f]$ is equal to the number of bipartite components of $\Gamma(f)$.*

Remark 2.9. It follows that all vertices of $P(d)$ are elements of $T(d)$, since if $f \in P(d)$ is a vertex then $\Gamma(f)$ has no bipartite component, so by Remark 2.1 and Remark 2.2 f must be an element of $T(d)$. We denote the vertices of $P(d)$ by $VT(d)$.

Remark 2.10. The elements of $T(d)$ are *minimal* in $P(d)$ in the sense that for each $f \in P(d)$ there is an $f' \in T(d)$ with $f'(x) \leq f(x)$ for all $x \in X$. For example, any vertex of $[f]$ is such an element. Furthermore, if for $f \in P(d)$ and $f' \in T(d)$ we have $f(x) \leq f'(x)$ for all $x \in X$, then

$$f'(x) = \max_{y \in X} \big(d(x,y) - f'(y)\big) \leq \max_{y \in X} \big(d(x,y) - f(y)\big) \leq f(x) \leq f'(x)$$

for all $x \in X$, so $f' = f$.

2.4 Coherent decompositions

Given a decomposition of a metric $d \in M(X)$ as a sum $d = d_1 + \cdots + d_k$ of metrics $d_1, \ldots, d_k \in M(X)$, the associated polytope of d trivially satisfies the inclusion

$$P(d) \supseteq P(d_1) + \cdots + P(d_k), \tag{2}$$

where

$$P(d_1) + \cdots + P(d_k) := \{f_1 + \cdots + f_k \mid f_1 \in P(d_1), \ldots, f_k \in P(d_k)\}.$$

If the above inclusion is in fact an equality, we say that $d = d_1 + \cdots + d_k$ is a *coherent decomposition* of the metric d, and we say that the sum $d_1 + \cdots + d_k$ is a *coherent sum*, and the set $\{d_1, \ldots, d_k\}$ is called a *coherent set*. A metric $d' \in M(X)$ is called a *coherent component* of d, which we will write $d' \lesssim d$, if there exists an $\alpha \in \mathbb{R}_+$ such that $d = \alpha d' + d''$ is a coherent decomposition of d (where $d'' := d - \alpha d'$).

Note that the relation $\lesssim$ is a preorder on $M(X)$. That is, $\lesssim$ is reflexive (since for example $d = \frac{1}{2}d + \frac{1}{2}d$ is a coherent decomposition of $d \in M(X)$), and transitive. For transitivity, let $d, d', d'' \in M(X)$ with $d' \lesssim d$ and $d'' \lesssim d'$. Then $d = \alpha d' + d_1$ and $d' = \alpha' d'' + d_2$ for some $\alpha, \alpha' \in \mathbb{R}_+$ and $d_1, d_2 \in M(X)$. Hence

$$d = \alpha(\alpha' d'' + d_2) + d_1 = \alpha\alpha' d'' + (\alpha d_2 + d_1)$$

is a coherent decomposition of d, and thus $d'' \lesssim d$, as claimed. For each $d \in M(X)$, we let

$$M(d) := \{d' \in M(X) \mid d' \lesssim d\}.$$

We can thus restate the definition of coherent decompositions according to the following proposition.

Proposition 2.11. *The decomposition $d = d_1 + \cdots + d_k$ of the metric $M(X)$ into the metrics $d_1, \ldots, d_k \in M(X)$ is coherent if and only if $d_1, \ldots, d_k \in M(d)$.*

Hence, the study of coherent decompositions of d is transformed to the study of the set $M(d)$. We will arrive at a nice description of the set $M(d)$, but for now we only have tools to work with coherent decompositions directly. The following proposition is our first step to this description.

Proposition 2.12. *For metrics $d, d_1, \ldots, d_k \in M(X)$ with $d = d_1 + \cdots + d_k$ the following statements are equivalent.*

(i) The decomposition $d = d_1 + \cdots + d_k$ is coherent,

(ii) for each $f \in P(d)$ there are elements $f_i \in P(d_i)$, $1 \leq i \leq k$ with $f = f_1 + \cdots + f_k$,

(iii) for each $f \in T(d)$ there are elements $f_i \in T(d_i)$, $1 \leq i \leq k$ with $f = f_1 + \cdots + f_k$,

(iv) for each $f \in VT(d)$ there are unique elements $f_i \in VT(d_i)$, $1 \leq i \leq k$ with $f = f_1 + \cdots + f_k$,

(v) for each $f \in P(d)$ there are elements $f_i \in P(d_i)$, $1 \leq i \leq k$ with

$$K(f) = \bigcap_{i=1}^{k} K(f_i).$$

Proof. First we note that (i) and (ii) are equivalent by the definition of coherent decomposition. Now assume (ii) holds, let $f \in P(d)$ and $f_i \in P(d_i)$, $1 \leq i \leq k$ with

$$f = f_1 + \cdots + f_k.$$

Thus (v) follows from Lemma 2.3. In particular we have $K(f) \subseteq K(f_i)$ for all $1 \leq i \leq k$, so if $K(f)$ is non-singular, then $K(f_1), \ldots, K(f_k)$ are all non-singular. Thus $f_i \in T(d_i)$ for all $1 \leq i \leq k$ by Remark 2.1, and so (ii) implies (iii). Furthermore, by Lemma 2.8 it follows that

$$\dim[f] \geq \dim[f_i] \tag{3}$$

for all $1 \leq i \leq k$. In particular if $\dim[f] = 0$ then $\dim[f_i] = 0$ for all $1 \leq i \leq k$, i.e. if $f \in VT(d)$ then $f_i \in VT(d_i)$ for all $1 \leq i \leq k$. For uniqueness, suppose

$$f = f_1 + \cdots + f_k = f_1' + \cdots + f_k'$$

for some $f_i \in VT(d_i)$, $1 \leq i \leq k$. Then $K(f) \subseteq K(f_i) \cap K(f_i')$ for all $1 \leq i \leq k$, i.e. the graph $\big(X, K(f_i) \cap K(f_i')\big)$ is non-singular and contains no bipartite component. Hence by Corollary 2.5 there is a face in $T(d_i)$ containing f_i and f_i', and by Corollary 2.4 the relative interior of f_i and f_i' is a vertex, i.e. $f_i = f_i'$, and so (ii) implies (iv).

Now, (ii) follows from (iii) by Remark 2.10, and (iii) follows from (iv) by Corollary 2.4. Finally, assume that (v) holds, and let $f \in VT(d)$ and $f_i \in P(d_i)$, $1 \leq i \leq k$ with

$$K(f) = \bigcap_{i=1}^{k} K(f_i).$$

Let $f' = f_1 + \cdots + f_k$. Then by Lemma 2.3

$$K(f') = \bigcap_{i=1}^{k} K(f_i) = K(f),$$

so since $f \in VT(d)$ we have $f' = f$, and thus (v) implies (iii). $\square$

Coherent decomopositions are interesting since they induce a decomposition of the tight span. In detail, we have the following situation. Let $d, d' \in M(X)$ be metrics with $d' \lesssim d$, and define a map $\psi^d_{d'} : \mathcal{F}(d) \to \mathcal{F}(d')$ by

$$\psi^d_{d'}([f]) := \{f' \in T(d') \mid K(f') \supseteq K(f)\}.$$

To see that $\psi^d_{d'}$ is well defined, note that for each $f \in P(d)$ there is an element $f' \in P(d')$ with $K(f) \subseteq K(f')$, by Proposition 2.12 (v), and hence the image of $[f]$ is a face of $P(d')$. Furthermore, by Corollary 2.5, $\psi^d_{d'}([f])$ is a face of $T(d')$ for all $f \in T(d)$. Also, $\psi^d_{d'}$ is surjective, since for each $f' \in P(d')$, for an element $f \in P(d)$ with $K(f)$ maximal with $K(f) \subseteq K(f')$ we have $\psi^d_{d'}([f]) = [f']$. Also, $\psi^d_{d'}$ preserves incidence, i.e. $[f] \subseteq [f']$ implies $\psi^d_{d'}([f]) \subseteq \psi^d_{d'}([f'])$, and by (3) we have

$$\dim[f] \geq \dim \psi^d_{d'}([f])$$

for all $f \in T(d)$. Finally, for each $x \in X$ we have

$$\psi_{d'}^{d}\big([\phi_d(x)]\big) = [\phi_{d'}(x)].$$

Studying the situation above, we see that the crucial fact that makes $\psi_{d'}^{d}$ well-defined is the fact that for each $f \in P(d)$ there is an $f' \in P(d')$ with $K(f) \subseteq K(f')$. Hence if this is true, there is an incidence-preserving map $\psi : \mathcal{F}(d) \to \mathcal{F}(d')$ with $\psi([\phi_d(x)]) = [\phi_{d'}(x)]$ for all $x \in X$. By the following theorem, this is also precisely what is needed for $d' \lesssim d$ to hold.

Theorem 2.13 (Variation of Theorem 4.1, [8])**.** *Let d and d' be two metrics on X. Then $d' \lesssim d$ if and only if for each $f \in P(d)$ there is an $f' \in P(d')$ such that $K(f) \subseteq K(f')$.*

Proof. The 'only if' part follows from Proposition 2.12 (v). For the 'if' part, we need to find an $\alpha \in \mathbb{R}_+$ such that $d'' := d - \alpha d'$ is a metric, and that for each $f \in VT(d)$ there are elements $f' \in VT(d')$ and $f'' \in VT(d'')$ with $f = f' + f''$.

By assumption, for each $f \in VT(d)$ there is an $f' \in VT(d')$ such that $K(f) \subseteq K(f')$. Hence, for each $x, y \in X$ with $f'(x) + f'(y) > d'(x, y)$ we have $f(x) + f(y) > d(x, y)$, so for all such $x, y \in X$ we have

$$\frac{f(x) + f(y) - d(x, y)}{f'(x) + f'(y) - d'(x, y)} > 0.$$

Since the set $VT(d) \cup VT(d')$ is finite, it follows that there is an $\alpha \in \mathbb{R}_+$ with

$$\frac{f(x) + f(y) - d(x, y)}{f'(x) + f'(y) - d'(x, y)} \geq \alpha$$

for all $f \in VT(d)$, $f' \in VT(d')$ with $K(f) \subseteq K(f')$ and $x, y \in X$ with $\{x, y\} \notin K(f')$. In particular, we have

$$f(x) + f(y) - d(x, y) \geq \alpha f'(x) + \alpha f'(y) + \alpha d'(x, y),$$

or equivalently,

$$(f - \alpha f')(x) + (f - \alpha f')(y) \geq (d - \alpha d')(x, y),$$

for all $f \in VT(d)$, $f' \in VT(d')$ and $x, y \in X$, with equality for all $\{x, y\} \in K(f)$ if $K(f') \subseteq K(f)$. Hence, if $d'' := d - \alpha d'$ is a metric, the result follows by Proposition 2.12 (iii). Thus we need to show that d'' satisfies the triangle inequality, i.e. that

$$d''(x,y) + d''(y,z) \geq d''(x,z)$$

for all $x, y, z \in X$. To see this, consider $f_y \in T(d)$, i.e. the element with $f_y(u) = d(y,u)$ for all $u \in X$, and let $f' \in V(d')$ with $K(f_y) \subseteq K(f')$. In particular, $\{y, y\} \in K(f')$, so $f'(y) = 0$, and thus $f'(u) = d'(y,u)$ for all $u \in X$. Hence

$$\begin{aligned} d''(x,z) &\leq (f_y - \alpha f')(x) + (f_y - \alpha f')(z) \\ &= f_y(x) - \alpha f'(x) + f_y(z) - \alpha f'(z) \\ &= d(x,y) - \alpha d'(x,y) + d(y,z) - \alpha d'(y,z) \\ &= (d - \alpha d')(x,y) + (d - \alpha d')(y,z) \\ &= d''(x,y) + d''(y,z). \end{aligned}$$

□

By the construction of α in the proof above, we have as an immediate consequence the following corollary.

Corollary 2.14 (Theorem 4.1, [8])**.** *Let d and d' be two metrics on X. Then $d = \alpha d' + d''$ is a coherent decomposition of d if and only if $0 \leq \alpha \leq \alpha^d_{d'}$, where*

$$\alpha^d_{d'} := \min_{f \in VT(d)} \left\{ \max_{f' \in VT(d')} \left\{ \min_{\{x,y\} \notin K(f')} \left\{ \frac{f(x) + f(y) - d(x,y)}{f'(x) + f'(y) - d'(x,y)} \right\} \right\} \right\}.$$

In particular, d' is a coherent component of d if and only if $\alpha^d_{d'} > 0$.

Corollary 2.15 (Corollary 3.5, [8])**.** *If $d = d_1 + \cdots + d_k$, $k \in \mathbb{N}$, is a coherent decomposition of $d \in M(X)$ into $d_1, \ldots, d_k \in M(X)$, then*

$$d' = \alpha_1 d_1 + \cdots + \alpha_k d_k$$

is a coherent decomposition of d' for all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_{\geq 0}$.

Proof. If $d' = \alpha_1 d_1 + \cdots + \alpha_k d_k$ is a coherent decomposition, then

$$\beta d' = \beta \alpha_1 d_1 + \cdots + \beta \alpha_k d_k$$

is a coherent decomposition for all $\beta \in \mathbb{R}_+$, so we may assume that $0 < \alpha_1 < 1$. We use induction on k. If $k = 1$ the statement is trivially true. So suppose it is true for $k = p - 1 \geq 1$ and let $\alpha_1, \ldots, \alpha_p \in \mathbb{R}_{\geq 0}$. Then by the induction hypothesis

$$d'' = \alpha_2 d_2 + \cdots + \alpha_p d_p$$

is a coherent decomposition of d''. Now,

$$d' = \alpha_1 d_1 + \cdots + \alpha_p d_p = \alpha_1 d_1 + d'',$$

so since $\alpha_1 < 1 \leq \alpha_{d_1}^{d'}$ the result follows by Corollary 2.14. $\square$

The value $\alpha_{d'}^{d}$ is called the *coherency index of* d *with respect to* d', and was defined in [8]. The coherency index provides a nice way to check whether or not a decomposition of a metric is coherent.

By Proposition 2.6 we see that $d' \lesssim d$ and $d \lesssim d'$ if and only if $d \sim d'$. In particular, $d \sim d'$ if and only if $M(d) = M(d')$. Hence $\lesssim$ induces a partial order on the types of metrics on $M(X)$, with $d' \lesssim d$ if d' can be obtained from d by possibly 'collapsing' some of d:s structure. If $d' \lesssim d$ but $d' \nsim d$ we write $d' \lnsim d$.

We are now ready to describe the $M(d)$ for $d \in M(X)$. Two natural classes of metrics arise from the partial order $\lesssim$, the minimal non-zero metrics, which we call the *prime* metrics, and the maximal metrics, which we call the *generic* metrics (the word *generic* being motivated by the previous paragraph). From now on, let Π be a fixed set of prime metrics on X, one of each type, and for each metric d on X let

$$\Pi(d) := \{p \in \Pi \mid p \lesssim d\}.$$

From Corollary 2.7 it follows that $\Pi(d)$ is finite for all metrics d on X. Also, it follows from Corollary 2.15 that

$$M(d) := \left\{ \sum_{p \in \Pi(d)} \alpha_p p \;\middle|\; \alpha_p \in \mathbb{R}_{\geq 0} \right\},$$

and thus $M(d)$ is the cone positively spanned by $\Pi(d)$.

Proposition 2.16. *Let d be a metric on X. Then d is in the relative interior of the cone $M(d)$. In particular, $d' \simeq d$ for some $d' \in M(X)$ if and only if d' is contained in the relative interior of $M(d)$, and $d' \lnsim d$ if and only if d' is contained in one of the proper faces of $M(d)$.*

Proof. Since $\alpha_p^d > 0$ for all $p \in \Pi(d)$, and since the set $\Pi(d)$ is finite there is an $\alpha \in \mathbb{R}_+$ with

$$d = d' + \sum_{p \in \Pi(d)} \alpha p$$

a coherent decomposition of d, where $d' \in M(d)$. $\square$

Proposition 2.16 gives us a way of characterizing the generic metrics in terms of maximal coherent subsets of Π.

Corollary 2.17. *The generic types of metrics on X correspond exactly to the maximal coherent subsets $P \subset \Pi$, i.e. the maximal subsets such that*

$$d = \sum_{p \in P} p$$

is a coherent sum.

3 Classifying the six point metrics

The aim of this paper was originally to classify the types of metrics when $|X| = 6$. However, during the course of this work, a classification was presented in [11] by Sturmfels and Yu. Their results are summarized in the following theorem.

Theorem 3.1 (Theorem 1, [11])**.** *There are* 194, 160 *generic types of metrics on six points, and they come in* 339 *symmetry classes.*

We have independently verified their results, using ideas presented in this paper. We searched for the maximal coherent subsets of Π, which by Corollary 2.17 correspond to the generic types of metric on X. It requires that the set Π is known.

The prime metrics on six points were characterized in [9], and there are 1, 235 of them, in 14 symmetry classes. Just testing each subset $P \subset \Pi$ if it is coherent or not is hence intractable, since the number of tests needed are in the order of $2^{1,235}$. To minimize the search space we used the fact that if $P \subset \Pi$ is a coherent set, then $\{p, p'\}$ must be a coherent set for all $p, p' \in P$. Hence we created a graph whose vertices are the prime metrics, and the edges are all pairs $p, p' \in \Pi$ with $\{p, p'\}$ a coherent set. We then searched for all maximal cliques in this graph, since each maximal coherent set must be contained in some such clique. Up to symmetry, there were 482 such cliques, with the maximal having size 25. It was then possible to search directly for maximal coherent sets. Our results agreed completely with the results of B. Sturmfels and J. Yu.

B. Sturmfels and J. Yu have set up an excellent web page where all the six point metrics are listed, along with images, at

`http://bio.math.berkeley.edu/SixPointMetrics`.

We used programs written in C++ and Objective Caml for our computations, which can be obtained by request from J. Kåhrström (through email, `johan.kahrstrom@math.uu.se`). The C++-programs use the package CDD++ by K. Fukuda for calculating the tight span of a metric. This package can be downloaded from

`http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html`.

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