Trees and Buildings

By Johan Kährström
Abstract

The tight span construction can be used to reconstruct a finite weighted tree from only using the distances between the leaves. It turns out that this construction can also be used on an object called a valuated matroid, yielding a very interesting geometrical object. It has been shown by W. Terhalle [11] that the tight span $T_M$ of a valuated matroid associated to a vector space over a discretely valuated field $F$ is a geometrical realization of the Euclidean building for the group $\text{SL}_n(F)$. In [2], F. Bruhat and J. Tits described the same building in terms of decomposable norms on the vector space $F^n$.

In this essay, we give a brief introduction to the theory on buildings, show why $T_M$ is a geometric realization of the building for the group $\text{SL}_n(F)$, and also show that the elements of the tight span of the valuated matroid described by W. Terhalle is precisely the set of decomposable norms of volume zero.
# Contents

Introduction ........................................ v

1 Simplicial complexes ................................ 1
   1.1 Simplicial complexes .............................. 1
   1.2 Abstract simplicial complexes .................... 4
   1.3 Flag complexes .................................. 5

2 Coxeter complexes ..................................... 9
   2.1 Coxeter groups .................................. 9
   2.2 Coxeter complexes ................................ 14

3 Buildings ............................................. 23
   3.1 Buildings ....................................... 23
   3.2 Strongly transitive groups ......................... 28
   3.3 The spherical building of \( \text{GL}_n(F) \) ............ 30
   3.4 The affine building of \( \text{SL}_n(F) \) ............... 30

4 The tight span of an \( \mathbb{R} \)-tree ................. 33
   4.1 \( \mathbb{R} \)-trees .................................. 33
   4.2 The tight span of a finite tree .................... 34
   4.3 The tight span of an infinite tree ................. 36

5 Valuated matroids ..................................... 39
   5.1 Valuated matroids ................................ 39
   5.2 The valuated matroid \( M_\alpha(F) \) ................. 44

6 Norms and the tight span ............................. 49
   6.1 Decomposable norms ................................ 49
   6.2 Relating decomposable norms and the tight span .... 52

A Discretely valuated fields ............................ 55
   A.1 Discrete valuations .............................. 55
   A.2 \( \mathcal{O} \)-lattices ................................ 57

Index of notation ...................................... 59

Bibliography ............................................ 61
Introduction

A phylogenetic tree is a tree representing the ancestry of a set of species. When constructing such trees, a natural problem arises: Given only the distances between the leaves of a weighted tree, is it possible to reconstruct the tree? The answer is yes, and one method for doing this is by constructing the **tight span** of the metric space consisting of the leaves and the distance between them.

It turns out that the tight span construction can be extended to infinite trees that have no leaves. That is, every branch extends infinitely far off towards infinity. Of course, in this case we have no metric on the infinite ends, but we can construct a distance-like function on the infinite ends, that is close enough to being a metric for the tight span construction to work.

Such an infinite tree, together with the distance-like function, is an example of a **valuated matroid**. The tight span of a general valuated matroid is not necessarily a tree-like structure, but a higher dimensional analogy.

It was shown in [11] by W. Terhalle that the tight span of a certain valuated matroid (denoted \( M_n(F) \)) obtained from the vector space \( F^n \), where \( F \) is a discretely valuated field, is a geometric realization of the **building** for the group \( \text{SL}_n(F) \). In [2], F. Bruhat and J. Tits describe another geometric realization of this building, that consists of the set of **decomposable norms** on the vector space \( F^n \).

The elements of the tight span described by W. Terhalle and decomposable norms have some striking similarities, which was pointed out by M. Ronan, and which we make more precise here.

Chapters 1-3 of this thesis gives a brief introduction to the theory on buildings, with emphasis on the theory needed to describe the building for the group \( \text{SL}_n(F) \). Chapter 4 describes the tight span construction for finite and infinite trees. Chapter 5 begins by introducing the concept of valuated matroids and describes the tight span of a valuated matroid and concludes by showing that the tight span of the valuated matroid \( M_n(F) \) obtained from the vector space \( F^n \) is a geometric realization of the building for \( \text{SL}_n(F) \). Most of the material in this chapter is taken from [11], by W. Terhalle. Finally, in Chapter 6 we look at the theory of norms on \( F^n \) and show that the elements of the tight span of \( M_n(F) \) is precisely the subset of decomposable norms on \( F^n \) having **volume zero**.

The appendix deals with discretely valuated fields, and any reader that is unfamiliar with these is strongly advised to read through the appendix before reading Chapters 3, 5 and 6, since discretely valuated fields are used extensively there.

Acknowledgment

I would like to thank my supervisor Vincent Moulton for all of his help, and in particular for giving me the opportunity to come to Uppsala in June and July 2002 to work on this thesis. I would also like to thank the Swedish Research Council (VR) for its funding during these months. Andreas Dress has also given helpful comments, which I am very grateful for.

I would also like to thank Stefan Borell, Rafal Czyż and especially Andreas Lind for their help, and for having the patience to listen to my ramblings on infinite trees and valuated matroids for the past year. Finally, I would like to thank Jörgen Boo for taking the time to read this thesis and acting as opponent.
INTRODUCTION
Chapter 1

Simplicial complexes

Simplicial complexes can be seen as a higher dimensional generalization of combinatorial graphs. Since the edges of a graph can be represented by lines, they are in a sense 1-dimensional. A simplicial complex can be seen as a graph with edges of possibly dimension higher than one.

1.1 Simplicial complexes

Definition 1.1. A simplicial complex $\Delta$ with vertex set $V$ is a set of finite subsets of $V$ called simplices satisfying

(SC1) for every singleton (called a vertex) $v \in V$, the set $\{v\}$ is in $\Delta$,

(SC2) if $A \in \Delta$, then $B \in \Delta$ for all $B \subseteq A$.

If $B \subseteq A$ for some $A, B \in \Delta$, then $B$ is said to be a face of $A$, written $B \leq A$. A simplex that is not a face of any other simplex is called a maximal simplex.

To simplify writing, we will sometimes say that an element $v \in V$ is in $\Delta$, where we of course mean that the set $\{v\}$ is in $\Delta$.

Example 1.2 If $V$ is any finite set, then $\Delta = \mathcal{P}(V)$, the powerset of $V$, is a simplicial complex.

Example 1.3 Consider a graph with vertex set $V$ and edge set $E$ consisting of a set of 2-subsets of $V$. Then $\Delta = V \cup E \cup \{\emptyset\}$ is a simplicial complex.

Example 1.4 Let $V = \{1, 2, 3, 4\}$ and

$$\Delta = \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}.$$ 

Then $\Delta$ is a simplicial complex. The elements of a simplicial complex can be ordered in a natural way, by set inclusion. In this case, the poset lattice of $\Delta$ is as follows:

Another way of visualizing $\Delta$ is to identify the singletons $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$ with four points in the plane, then identify any simplex of $\Delta$ with the convex hull of the points it contains (see Figure
CHAPTER 1. SIMPLICIAL COMPLEXES

Definition 1.5. A simplex of cardinality \( r \) is said to have rank \( r \), and dimension \( r - 1 \). A simplex of dimension \( n \) is called an \( n \)-simplex. The empty set is said to have rank 0 and dimension \(-1\).

By looking at Example 1.4, it should be obvious that an \( n \)-simplex can be identified with a geometrical \( n \)-simplex, i.e., the convex hull of \((n + 1)\) points in an \( n \)-dimensional real vector space, not lying in a common hyperplane. This is a loose definition of what is called a geometric realization of the \( n \)-simplex. The geometric realization of a simplicial complex is obtained by first forming the geometric realization of all the maximal simplices, and then ‘gluing’ them together by identifying common points. This is all a little vague, so let us construct the geometric realization of the simplicial complex of Example 1.4 to get a better feeling for what is happening.

There are two maximal simplices, \( \{1, 2, 3\} \) and \( \{2, 3, 4\} \), so first form the geometric realization of these two, which gives us two triangles in \( \mathbb{R}^2 \). Note that the singleton \( \{2\} \) gives rise to two points, one for each maximal simplex. We will call these points \( 2_1 \) and \( 2_2 \) to be able to make this distinction. The same goes for the singleton \( \{3\} \), which gives rise to the two points \( 3_1 \) and \( 3_2 \). Now identify the two points corresponding to the set \( \{2\} \) with each other, and do the same for the points corresponding to the set \( \{3\} \). Also, identify the points in the convex hull of \( 2_1 \) and \( 3_1 \) with the points of the convex hull of \( 2_2 \) and \( 3_2 \) in the natural way, which gives us the geometric realization of \( \Delta \) (see Figure 1.2).

Note that the geometric realization of a simplicial complex whose maximal simplices have dimension \( n \) is generally not embeddable in \( \mathbb{R}^n \). For example, the geometric realization of the simplicial complex

\[
\Delta = \{ \{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \emptyset \}
\]

is not embeddable in \( \mathbb{R}^2 \), but it is embeddable in \( \mathbb{R}^3 \) (see Figure 1.3).

Definition 1.6. A subcomplex \( \Delta' \) of a simplicial complex \( \Delta \) with vertex set \( V \) is a subset of \( \Delta \) which is a simplicial complex in its own right.

Every simplex \( A \) of a simplicial complex \( \Delta \) can be seen as a subcomplex of \( \Delta \), by letting the elements of this subcomplex be the subsets of \( A \) (which are all in \( \Delta \)). We will denote this subcomplex by \( \Delta_{\leq A} \).

Example 1.7. Let \( \Delta \) be as in Example 1.4 and let \( A = \{1, 2, 3\} \). Then the subcomplex \( \Delta_{\leq A} \) is

\[
\Delta_{\leq A} = \{ \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset \}.
\]
1.1. SIMPLICIAL COMPLEXES

Figure 1.3: The geometric realization of $\Delta$ is not embeddable in $\mathbb{R}^2$, but it is embeddable in $\mathbb{R}^3$.

$\Delta : \bullet \bullet \bullet \bullet \bullet \bullet \bullet$

$\Delta' : \bullet \bullet \bullet \bullet \bullet \bullet \bullet$

Figure 1.4: The two simplices of Example 1.8.

**Example 1.8** Let $V = \mathbb{Z}$ and $\Delta = \{ \{n\} \mid n \in \mathbb{Z}\} \cup \{\{n, n + 1\} \mid n \in \mathbb{Z}\} \cup \emptyset$ (see Figure 1.4). Then $\Delta$ is a simplicial complex. Every integer is a 0-simplex, and every set $\{n, n + 1\}$ is a 1-simplex which can be identified with the line between $n$ and $n + 1$ on the real line. The set

$$\Delta' = \{\{n\} \mid n \in \mathbb{Z}\} \cup \{\{2n, 2n + 1\} \mid n \in \mathbb{Z}\} \cup \emptyset$$

is a subcomplex of $\Delta$.

**Definition 1.9.** A chamber complex is a simplicial complex $\Delta$ such that:

(CC1) All maximal simplices have the same rank. The maximal simplices of a chamber complex are called chambers. We say that two chambers are adjacent if they share a face of co-dimension 1.

(CC2) For any two chambers $C, C' \in \Delta$ there is a sequence of chambers $C_1, \ldots, C_r$ such that $C = C_1$, $C' = C_r$ and $C_i$ is adjacent to $C_{i+1}$ for all $1 \leq i \leq r - 1$. Such a sequence of chambers is said to be a gallery from $C$ to $C'$.

The rank of the chamber complex $\Delta$ is the common rank of the chambers of $\Delta$. A chamber complex is said to be thin if every simplex of co-dimension 1 is a face of exactly two chambers. If every simplex of co-dimension 1 is a face of at least three chambers, then the chamber complex is said to be thick.

The simplicial complex of Example 1.4 is a chamber complex, as is $\Delta$ in Example 1.8. The simplicial complex of Example 1.3 is a chamber complex if and only if the graph is connected (otherwise it would not satisfy CC2).

A labeling of the chamber complex $\Delta$ by a set $I$ is a map $\lambda$ from the vertices of $\Delta$ to $I$ such that $\lambda$ restricted to any chamber of $\Delta$ is a bijection. For example, a labeling of the chamber complex of Example 1.4 by the set $\{a, b, c\}$ is the map $\lambda(1) = \lambda(4) = a, \lambda(2) = b, \lambda(3) = c$. A labeling of the vertices of $\Delta$ can be naturally extended to the whole of $\Delta$, by defining

$$\lambda(A) = \{\lambda(a) \mid a \in A\}.$$

Figure 1.5: A gallery connecting $C$ with $C'$. 
CHAPTER 1. SIMPLICIAL COMPLEXES

Figure 1.6: Example 1.11.

We say that $\lambda(A)$ is the type of $A$.

Two adjacent chambers $C$ and $C'$ share all but one vertex, and hence the two vertices of $C \triangle C'$ (the symmetric difference of $C$ and $C'$) must have the same type. We say that the two chambers are $i$-adjacent, where $i$ is the type of the two non-common vertices. Equivalently, two chambers are $i$-adjacent if they share a co-dimension 1 face of type $I \setminus \{i\}$.

1.2 Abstract simplicial complexes

Every simplicial complex can be seen as a poset, ordered by inclusion. A poset that is isomorphic to the poset of a simplicial complex contains all the information of the simplicial complex, and can thus in a sense be seen as a simplicial complex itself. Thus the following extension of Definition 1.1 is natural.

Definition 1.10. A poset is said to be an abstract simplicial complex if it is isomorphic to the poset of some simplicial complex as defined in Definition 1.1.

In this essay, we will make no distinction between simplicial complexes as defined by Definition 1.1 and abstract simplicial complexes, but call both just simplicial complexes.

Example 1.11 Let $p_1, p_2$ and $p_3$ be three points in $\mathbb{R}^2$, not on a common line, and let $l_1$ be the line between $p_1$ and $p_2$, $l_2$ be the line between $p_2$ and $p_3$ and let $l_3$ be the line between $p_1$ and $p_3$ (see Figure 1.6). Finally, let

$$T = \{l_1, l_2, l_3, p_1, p_2, p_3, \emptyset\},$$

and order $T$ by inclusion (where the elements are seen as subsets of $\mathbb{R}^2$). Let $\Delta$ be the simplicial complex defined in Example 1.18, i.e. the boundary of the simplex $\{1, 2, 3\}$. The map $\phi : T \rightarrow \Delta$ defined by

$$l_1 \mapsto \{1, 2\} \quad l_2 \mapsto \{2, 3\} \quad l_3 \mapsto \{1, 2\}$$
$$p_1 \mapsto \{1\} \quad p_2 \mapsto \{2\} \quad p_3 \mapsto \{3\}$$
$$\emptyset \mapsto \emptyset$$

is easily seen to be a poset isomorphism, and hence $T$ is a simplicial complex.

Example 1.12 Consider the geometric realization of a simplicial complex, and let $\Delta$ be the set of all convex hulls of the simplices of the simplicial complex. Ordering $\Delta$ by inclusion, it is easy to see that $\Delta$ is isomorphic to the original simplicial complex.

It is not obvious at first sight whether or not a poset is a simplicial complex. The following proposition characterizes the simplicial complexes among posets.
1.3. FLAG COMPLEXES

Proposition 1.13. A poset $\Delta$ is a simplicial complex if and only if

(i) Any two elements $A, B \in \Delta$ have a greatest lower bound in $\Delta$.

(ii) For any $A \in \Delta$ the poset of faces of $A$ is isomorphic to the poset of subsets of $\{1, \ldots, r\}$ for some $r \geq 0$.

Proof. We will start by showing the ‘only if’ part. Suppose that the poset $\Delta$ is isomorphic to a simplicial complex by the map $\phi$. Then for any two elements $A, B \in \Delta$ it is obvious that $\phi^{-1}(\phi(A) \cap \phi(B))$ is a lower bound in $\Delta$ for $A$ and $B$, and this is by necessity the greatest lower bound of $A$ and $B$. Hence $\Delta$ satisfies (i). That $\Delta$ satisfies part (ii) is obvious.

To show the ‘if’ part, define the rank of the $A \in \Delta$ to be the unique integer $r$ such that $A$ satisfies (ii). Now, let $V$ be the set of rank 1 elements of $\Delta$, and for each $A \in \Delta$ define $A'$ to be

$$A' = \{v \in V | v \leq A\}.$$  

The set of such elements is a simplicial complex, to which $\Delta$ is isomorphic. $\square$

1.3 Flag complexes

Definition 1.14. Let $P$ be a set with some binary relation that is both reflexive and symmetric. A flag in $P$ is a set of pairwise related elements, and the flag complex of $P$ is the set of finite flags of $P$, ordered by set inclusion.

That the flag complex is indeed a simplicial complex is immediate: Each singleton of $P$ is a ‘set of pairwise related elements’, and hence the vertex set of the flag complex of $P$ is simply $P$. Also, any subset of a set of related elements is a set of related elements and hence (SC1) and (SC2) are satisfied.

When we speak of the flag complex of a simplicial complex, then we assume that the relation is defined only on the non-empty simplices, and that two simplices are related if and only if one is a face of the other. The reason for excluding the empty set, is that it would be related to all other simplices, and hence would be redundant.

Example 1.15. Let $\Delta$ be a 1-simplex. As we have seen, $\Delta$ can be identified with two points, and the line connecting them. Label the points 1 and 2 (see Figure 1.7(a)).

The flag complex of $\Delta$ is as follows:

```
{1} ⊆ {1, 2}   {2} ⊆ {1, 2}

\{1\}           \{1, 2\}         \{2\}
\downarrow          \downarrow          \downarrow
\{1\}           \{1, 2\}         \{2\}
```

This is the simplicial complex of two 1-simplices sharing a common vertex, i.e. the point/line incidence of Figure 1.7(b). Hence the line has been divided into two parts.

![Figure 1.7](image)

Figure 1.7: (a) The simplicial complex $\Delta$ of Example 1.15, and its flag complex (b).

Example 1.16. Let $\Delta$ be a 2-simplex, with vertices 1, 2 and 3. The flag complex of $\Delta$ is shown in Figure 1.8. For readers familiar with the concept of barycentric subdivisions, we note that in general, if $P$ is a poset of simplices of a simplicial complex $\Delta$, then the flag complex of $P$ is the
CHAPTER 1. SIMPLICIAL COMPLEXES

Example 1.17 Let \( \Delta \) be a chamber complex with vertex set \( V \). Define a relation on \( V \) by saying that two elements are related if they are faces of a common chamber of \( \Delta \). Hence every set of pairwise related elements of \( V \) is an element of \( \Delta \), and vice versa. Thus the flag complex of \( V \) with this relation is precisely \( \Delta \).

Example 1.18 The boundary of an \( n \)-simplex with vertex set \( V \) is the simplicial complex \( \Delta = \mathcal{P}(V) \setminus V \). Now consider the boundary of the 2-simplex with vertex set \( \{1, 2, 3\} \). Its flag complex is shown in Figure 1.9. We see that all maximal simplices of \( \Delta \) have rank 2, and that all maximal simplices can be connected by a gallery. Hence \( \Delta \) is a chamber complex. Moreover, if we label the vertices of \( \Delta \) by their cardinality, we get a labeling of \( \Delta \). For example, \( \lambda(\{2\}) = 1 \) and \( \lambda(\{1, 3\}) = 2 \).

This holds in the general case as well: The flag complex of the boundary of an \( n \)-simplex is a rank \( n \) chamber complex, where a labeling on the vertices is given by the cardinality of the vertex.

Example 1.19 Define a relation on \( \mathbb{Z}^n \) by saying that two elements \( \mathbf{x} \) and \( \mathbf{y} \) are related if either \( \mathbf{x} = \mathbf{y} + R \) or \( \mathbf{y} = \mathbf{x} + R \), where

\[
R = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid x_i \in \{0, 1\}\}.
\]

For example, in \( \mathbb{Z}^2 \), the elements related to \( (0, 0) \) are

\[(0, 0), (0, 1), (1, 0), (1, 1), (0, -1), (-1, 0), (-1, -1)\].

See Figure 1.10. Define a labeling \( \lambda \) on \( \mathbb{Z}^n \) by

\[
\lambda(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \pmod{n+1}.
\]
Now let \( \Delta \) be the flag complex of this structure. By looking at Figure 1.10, it should be obvious that this is a chamber complex when \( n = 2 \). In the general case, let

\[
\begin{align*}
    a_1 &= (1,0,\ldots,0), \\
    a_2 &= (1,1,0,\ldots,0), \\
    &\vdots \\
    a_n &= (1,\ldots,1).
\end{align*}
\]

For any simplex \( A \) of \( \Delta \), there is a ‘minimal’ element \( x \) of \( A \) and a set \( I \subseteq \{1,\ldots,n\} \) such that

\[
A = \{ x + \sigma(a_i) \mid \sigma \in S_n, i \in I \},
\]

where \( S_n \) acts on a vector by permuting coordinates. It follows that the maximal simplices all have the same rank, which is \( n + 1 \). We will not show that all maximal simplices can be connected by a gallery, but this will follow by looking at Example 2.17. Also, it is obvious that \( \lambda \) restricted to any maximal simplex is a bijection, and hence \( \lambda \) is a labeling of \( \Delta \).
Chapter 2

Coxeter complexes

In this section we introduce a class of simplicial complexes called Coxeter complexes, which are fundamental to the study of buildings. Coxeter complexes have a lot of nice geometric properties, but we will only look at a small fraction of these in this thesis. For more information on this, see [1, 3, 10].

2.1 Coxeter groups

Definition 2.1. A group \( W \) is a Coxeter group if there is a set \( S = \{s_1, \ldots, s_n\} \subseteq W \) such that \( W \) admits the presentation

\[
W = \langle S \mid (s_is_j)^{m_{ij}} = 1 \text{ for all } i, j \rangle,
\]

where \( m_{ij} = m_{ji} \in \{2, 3, \ldots\} \cup \{\infty\} \) if \( i \neq j \) and \( m_{ii} = 1 \) for all \( i \).

Example 2.2 The group \( G = \{1, -1\} \) under multiplication is a Coxeter group, since it is generated by the set \( S = \{-1\} \) which is of order 2 and has the presentation \( G = \langle -1 \mid (-1)^2 = 1 \rangle \).

Example 2.3 Consider \( D_{2n} \), the dihedral group of order \( 2n \). This group is generated by two reflections \( \rho_1 \) and \( \rho_2 \) whose fixed lines are at an angle of \( \pi/n \), and it has the presentation

\[
D_{2n} = \langle \rho_1, \rho_2 \mid \rho_1^2 = \rho_2^2 = (\rho_1\rho_2)^n = 1 \rangle.
\]

Hence \( D_{2n} \) is a Coxeter group.

Example 2.4 The infinite dihedral group

\[
D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle
\]

is obviously a Coxeter group.

Example 2.5 Consider \( S_n \), the symmetric group on \( n \) letters, and let \( S = \{s_i \mid 1 \leq i < n\} \), where \( s_i \) is the transposition \( (i \ i+1) \), \( 1 \leq i < n \). Certainly, \( \langle S \rangle \), the subgroup of \( S_n \) generated by the set \( S \), is a subgroup of \( S_n \). To prove the reverse inequality, since \( S_n \) is generated by the set of transpositions, it suffices to show that every transposition can be written as a product of elements of \( S \). To see this, let \( \tau_{ij} \) be the transposition swapping \( i \) and \( j \). Then (assuming without loss of generality that \( i < j \))

\[
\tau_{ij} = s_is_{i+1} \cdots s_{j-2}s_{j-1}s_{j-2} \cdots s_{i+1}s_i.
\]
Define a map $\sigma \in S_{n+1}$. An easy way of thinking of this action of $\mathbb{R}$ is to think of every element $x = (x_1, \ldots, x_{n+1})$ by

$$\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n+1)}).$$

Now, let $B = \{b_1, \ldots, b_n\}$ be a basis of $\mathbb{R}^n$ such that $\|b_i\| = 1$ and the angle between any two basis vectors is $\frac{2\pi}{n}$. That is, the dot product satisfies

$$b_i \cdot b_j = \begin{cases} 1 & \text{if } i = j \\ \frac{1}{2} & \text{if } i \neq j \end{cases}.$$

Define a map $\psi$ from the subspace

$$H = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \left| \sum_{i=1}^{n+1} x_i = 0 \right. \right\}$$

of $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ by

$$\psi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n),$$

where the coordinates in $\mathbb{R}^{n+1}$ are given with respect to the standard basis and the coordinates in $\mathbb{R}^n$ are given with respect to the basis $B$. $\psi$ is obviously a homomorphism (it is linear), and it is a bijection since the coordinate $x_{n+1}$ of a point in $H$ is completely determined by the coordinates $x_1, \ldots, x_n$ since $x_{n+1} = -x_1 - \cdots - x_n$.

Let $S_{n+1}$ act on $\mathbb{R}^n$ by

$$\sigma(x) = \psi \circ \sigma \circ \psi^{-1}(x).$$

An easy way of thinking of this action of $S_{n+1}$ on $\mathbb{R}^n$ is to think of every element $\mathbb{R}^n$ as having an ‘invisible’ $(n+1)$th coordinate whose value is $-x_1 - \cdots - x_n$. For example, in $\mathbb{R}^2$ we have $(13)(x_1, x_2) = (-x_1 - x_2, x_2)$. 

Clearly $s_i^2 = 1$ for all $i, 1 \leq i < n$. Also, if $j \neq i + 1$ and $i \neq j + 1$ then $(s_is_j)^2 = (s_js_i)^2 = 1$. Finally, $s_is_{i+1}$ is the cycle $(i \ i + 1 \ i + 2)$, so $(s_is_{i+1})^3 = 1$.

Thus $S_n = \langle S | s_i^2 = (s_is_{i+1})^3 = (s_is_j)^2 = 1 \rangle$, and so $S_n$ is a Coxeter group.
As we saw in Example 2.5, $S_{n+1}$ is generated by the transpositions $s_1 = (12), \ldots, s_n = (n \; n+1)$, and we will now look at how these act on $\mathbb{R}^n$.

The transposition $s_1$ maps the point $(x_1, x_2, x_3, \ldots, x_n)$ to $(x_2, x_1, x_3, \ldots, x_n)$. This linear transformation is given by the matrix

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \ddots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}.$$ 

The eigenvalues of this linear transformation are the solutions to the equation $0 = \det(\lambda I - A) = (\lambda^2 + 1)(\lambda - 1)^{n-2} = (\lambda + 1)(\lambda - 1)^{n-1}$, and are thus $1$ and $-1$. The $1$-eigenspace is the hyperplane satisfying $x_1 = x_2$ which is spanned by the vectors $(b_1 + b_2), b_3, \ldots, b_n$ and the $(-1)$-eigenspace is the line satisfying $x_1 = -x_2, x_3 = \cdots = x_n = 0$, and this line is spanned by the vector $(b_1 - b_2)$.

Also, the $(-1)$-eigenspace is orthogonal to the $1$-eigenspace, since

$$(b_1 + b_2) \cdot (b_1 - b_2) = b_1 \cdot b_1 + b_2 \cdot b_2 - b_1 \cdot b_2 - b_2 \cdot b_1 = 1 + \frac{1}{2} - \frac{1}{2} - 1 = 0$$

and for $j > 2$ we have

$$b_j \cdot (b_1 - b_2) = b_j \cdot b_1 - b_j \cdot b_2 = \frac{1}{2} - \frac{1}{2} = 0.$$ 

This means that the action of $s_1$ is simply reflection through the plane $x_1 = x_2$. Similar calculations show that the action of $s_i, 1 \leq i < n$ is reflection through the plane $x_i = x_{i+1}$ and the action of $s_n$ is reflection through the plane $x_n = -x_1 - \cdots - x_n$.

We claimed that this action would be the group of symmetries of an $n$-simplex. One such simplex is the convex hull of the points in $\mathbb{R}^n$ given by

$$p_1 = \left(\frac{-n}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$$

$$p_2 = \left(\frac{1}{n+1}, \frac{-n}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$$

$$\vdots$$

$$p_n = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}, -\frac{n}{n+1}\right)$$

$$p_{n+1} = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right).$$

It is easily seen that for $\sigma \in S_{n+1}$, $\sigma$ acts on these points by $\sigma(p_i) = p_{\sigma(i)}$. Also, if $i < j \neq n + 1$ then

$$\|p_i - p_j\|^2 = \left\|\left(0, \ldots, 0, \frac{-n + 1}{n + 1}, 0, \ldots, 0, \frac{n + 1}{n + 1}, 0, \ldots, 0\right)\right\|^2 = b_i \cdot b_i - b_i \cdot b_j - b_i \cdot b_j + b_j \cdot b_j = 1 - \frac{1}{2} - \frac{1}{2} + 1 = 1,$$

and if $i < n + 1$ then

$$\|p_i - p_{n+1}\|^2 = \left\|\left(0, \ldots, 0, \frac{n + 1}{n + 1}, 0, \ldots, 0\right)\right\|^2 = b_j \cdot b_j = 1,$$

so the points are equidistant. Hence $S_{n+1}$ acts as the group of symmetries of an $n$-simplex in $\mathbb{R}^n$. Note also that an element of $S_{n+1}$ acts as a reflection if and only if it is a transposition.

If we restrict the action of $S_{n+1}$ to $\mathbb{Z}^n$ (that is, the $\mathbb{Z}$-module spanned by the basis $B$), then it is obvious that this action is closed (as it just permutes the coordinates, which are all integers). Also, for any $\sigma \in S_{n+1}$, the action of $\sigma$ on $\mathbb{Z}^n$ is an automorphism of $\mathbb{Z}^n$ and hence we can form
the semi-direct product $\mathbb{Z}^n \rtimes S_{n+1}$. An element $w \in \mathbb{Z}^n \rtimes S_{n+1}$ will be written $[x, \sigma]$, where $x \in \mathbb{Z}^n$ and $\sigma \in S_{n+1}$. Thus multiplication of two elements $[x, \sigma], [y, \tau] \in \mathbb{Z}^n \rtimes S_{n+1}$ is given by

$$[x, \sigma][y, \tau] = [x + \sigma(y), \sigma\tau].$$

We will now show (by some rather messy computations) that $\mathbb{Z}^n \rtimes S_{n+1}$ admits the presentation

$$\mathbb{Z}^n \rtimes S_{n+1} = \langle \tilde{s}_0, \ldots, \tilde{s}_n | (\tilde{s}_i \tilde{s}_j)^{m_{ij}} = 1 \rangle,$$

where

$$m_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
3 & \text{if } |i - j| = 1 \text{ or } |i - j| = n, \\
2 & \text{otherwise}.
\end{cases}$$

and hence is a Coxeter group.

**Example 2.6** Consider the group $\mathbb{Z}^n \rtimes S_{n+1}$. Let $\tilde{s}_i = [0, (i, i+1)]$, $1 \leq i \leq n$, and let

$$\tilde{s}_0 = [(-1,0,\ldots,0),(1\ n+1)].$$

First, we show $\langle \tilde{s}_0, \ldots, \tilde{s}_n \rangle = \mathbb{Z}^n \rtimes S_{n+1}$. It follows from Example 2.5 that $\tilde{s}_1, \ldots, \tilde{s}_n$ generate the set of elements of the form $[0, s], s \in S_n$. Also

$$[0,(1\ n+1)]\tilde{s}_0 = [(1,0,\ldots,0),1],$$

so $[(z,0,\ldots,0),1] \in \langle \tilde{s}_0, \ldots, \tilde{s}_n \rangle, z \in \mathbb{Z}$. Also, for $z \in \mathbb{Z}$ we have

$$[0,(1\ i+1)][(z,0,\ldots,0),1][0,(1\ i+1)] = [z_i,1],$$

where $z_i = (0,\ldots,z,0,\ldots,0)$ with the $z$ in the $i$'th coordinate. It follows that $\langle \tilde{s}_0, \ldots, \tilde{s}_n \rangle = \mathbb{Z}^n \rtimes S_{n+1}$.

We have that $\tilde{s}_j^2 = 1$ for all $j$, and it follows from Example 2.5 that for $1 \leq i, j \leq n$ we have

$$(\tilde{s}_i \tilde{s}_j)^3 = 1 \text{ if } |i - j| = 1 \text{ and } (\tilde{s}_i \tilde{s}_j)^2 = 1 \text{ if } |i - j| > 1.$$

For $1 < i < n$ we have

$$\tilde{s}_i \tilde{s}_0 = [0, s_i][(-1,0,\ldots,0),(1\ n+1)] = [(-1,0,\ldots,0),(i\ i+1)(1\ n+1)],$$

so

$$(\tilde{s}_i \tilde{s}_0)^2 = [(-1,0,\ldots,0),(i\ i+1)(1\ n+1)][(-1,0,\ldots,0),(i\ i+1)(1\ n+1)]$$

$$= [(-1,0,\ldots,0)+ (1,0,\ldots,0),1] = [(0,\ldots,0),1] = 1.$$

The only cases left are $\tilde{s}_1 \tilde{s}_0$ and $\tilde{s}_n \tilde{s}_0$. For $\tilde{s}_1 \tilde{s}_0$ we have

$$\tilde{s}_1 \tilde{s}_0 = [0, s_1][(-1,0,\ldots,0),(1\ n+1)] = [(0,-1,0,\ldots,0),(n+1\ 2\ 1)],$$

so

$$(\tilde{s}_1 \tilde{s}_0)^2 = [(0,-1,0,\ldots,0),(n+1\ 2\ 1)][(0,-1,0,\ldots,0),(n+1\ 2\ 1)]$$

$$= [(0,-1,0,\ldots,0)+(1,0,\ldots,0),(1\ n+1)]$$

$$= [(-1,0,\ldots,0),(1\ n+1)]$$

and

$$(\tilde{s}_1 \tilde{s}_0)^3 = [(-1,0,\ldots,0),(1\ n+1)][(0,-1,0,\ldots,0),(n+1\ 2\ 1)]$$

$$= [(-1,0,\ldots,0)+(1,0,\ldots,0),1] = [0,1] = 1.$$

Finally,

$$\tilde{s}_n \tilde{s}_0 = [0, s_n][(-1,0,\ldots,0),(1\ n+1)] = [(-1,0,\ldots,0,1),(n+1\ n\ 1)],$$

so

$$(\tilde{s}_n \tilde{s}_0)^2 = [(-1,0,\ldots,0,1),(n+1\ n\ 1)][(-1,0,\ldots,0,1),(n+1\ n\ 1)]$$

$$= [(-1,0,\ldots,0,1)+(0,\ldots,0,1),(1\ n+1)]$$

$$= [(-1,0,\ldots,0),(1\ n+1)]$$
2.1. COXETER GROUPS

and

\[(s_n s_0)^3 = [(−1, 0, . . . , 0, 1), (n+1 \ n 1)][(−1, 0, . . . , 0), (1 \ n \ n+1)]
\hspace{1cm} = [(−1, 0, . . . , 0, 1) + (1, 0, . . . , 0, −1), 1] = [0, 1] = 1.\]

It follows that \(\mathbb{Z}^n \rtimes S_{n+1}\) admits the presentation as claimed above.

The following lemma shows a very important property of the Coxeter groups. The proof is taken from [10], page 10.

**Lemma 2.7.** Let \(W\) be a Coxeter group as in Definition 2.1, and let \(S'\) be a subset of \(S\). Then

(i) The element \(s_i s_j\) has order \(m_{ij}\).

(ii) If \(s_i \in \langle S' \rangle\) then \(s_i \in S'\).

**Proof.** (i) In this proof, we define \(\frac{1}{m_{ij}} = 0\). Let \(V\) be a real vector space with a basis \(B = \{b_1, . . . , b_n\}\) and define a symmetric bilinear form on \(V\) by putting

\[\langle b_i, b_j \rangle = −\cos \frac{π}{m_{ij}}.\]

Note that \(\langle b_i, b_j \rangle = −1\) if \(m_{ij} = \infty\) and \(\langle b_i, b_i \rangle = 1\) for all indices \(1 \leq i \leq n\). For each \(i\), define a linear transformation \(r_i\) by

\[r_i(v) = v − 2 \langle v, b_i \rangle b_i,\] for all \(v \in V,\]

and let \(G\) be the subgroup of \(GL(V)\) generated by the set \(\{r_i \mid 1 \leq i \leq n\}\). For \(1 \leq i, j \leq n, i \neq j\), let \(V_{ij}\) be the subspace spanned by \(b_i\) and \(b_j\), and let \(V_{ij}^\perp\) be the orthogonal complement of \(V_{ij}\) (with respect to the bilinear form).

We will start by showing that given \(i \neq j, 1 \leq i, j \leq n\), then \(V_{ij} + V_{ij}^\perp = V\) if \(m_{ij} \neq \infty\). So, let \(u\) be any vector in \(V\). We want to find a vector \(v \in V_{ij}\) and a vector \(w \in V_{ij}^\perp\) such that \(u = v + w\). If we can find a vector \(v \in V_{ij}\) that satisfies

\[\langle u, b_i \rangle = \langle v, b_i \rangle\] (2.1)

and

\[\langle u, b_j \rangle = \langle v, b_j \rangle\] (2.2)

then putting \(w = u − v\) we have \(\langle w, b_i \rangle = \langle u, b_i \rangle − \langle v, b_i \rangle = 0\) and \(\langle w, b_j \rangle = \langle u, b_j \rangle − \langle v, b_j \rangle = 0\), and so \(w \in V_{ij}^\perp\). So suppose \(v = k_i b_i + k_j b_j\) satisfies Equation (2.1) and Equation (2.2). That is, \(k_i\) and \(k_j\) satisfy

\[\langle u, b_i \rangle = \langle v, b_i \rangle = \langle k_i b_i, b_i \rangle + \langle k_j b_j, b_i \rangle = k_i + k_j \langle b_i, b_j \rangle\]

and

\[\langle u, b_j \rangle = \langle v, b_j \rangle = \langle k_i b_i, b_j \rangle + \langle k_j b_j, b_j \rangle = k_1 \langle b_i, b_j \rangle + k_2.\]

This linear system is solvable if the matrix

\[
\begin{bmatrix}
1 & \langle b_i, b_j \rangle \\
\langle b_i, b_j \rangle & 1
\end{bmatrix}
\]

is invertible, which is the case if and only if \(\langle b_i, b_j \rangle \neq \pm 1\), i.e., if and only if \(m_{ij} \neq 1\) and \(m_{ij} \neq \infty\). Hence we have shown that if \(m_{ij} \neq \infty\) then \(V_{ij} + V_{ij}^\perp = V\).

Now, let’s consider the action of \(r_i\) and \(r_j\) on \(V_{ij}\), \(i \neq j, 1 \leq i, j \leq n\). If \(m_{ij} \neq \infty\), then we can identify \(V_{ij}\) equipped with the bilinear form above with \(\mathbb{R}^2\) together with the usual dot product. In this setting, the two vectors \(b_i\) and \(b_j\) are unit vectors, at angle \(π − \frac{π}{m_{ij}}\). Also, the action of \(r_i\) and \(r_j\) is simply reflection through the line perpendicular to \(b_i\) and \(b_j\) respectively. Hence, the action of \(r_i r_j\) is rotation by the angle \(2(π − \frac{π}{m_{ij}}) = \frac{2π}{m_{ij}}\). Hence \(r_i r_j\) has order \(m_{ij}\) on \(V_{ij}\). On the other hand, if \(m_{ij} = \infty\), then a straightforward calculation shows that \(r_i r_j\) is the linear transformation given by the matrix

\[
\begin{bmatrix}
3 & 2 \\
2 & −1
\end{bmatrix}.
\]
This linear transformation has order \( \infty \). Hence, on \( V_{ij} \), \( r_i r_j \) has order \( m_{ij} \). It is obvious that both \( r_i \) and \( r_j \) are the identity on \( V_{ij}^\perp \). So, the order of \( r_i r_j \) on \( V \) is equal to \( m_{ij} \).

Now, let \( F \) be the free group on the set \( \{x_1, \ldots, x_n\} \) and let \( N \) be the smallest normal subgroup of \( F \) containing \( (x_ix_j)^{m_{ij}} \) for all \( i, j \) such that \( m_{ij} < \infty \). Then, by the definition of \( N \), we have \( F/N \cong W \), where \( x_i N = s_i \). Define a homomorphism \( \phi : F \to G \) by \( x_i \mapsto r_i \). For all \( i, j \) with \( m_{ij} < \infty \), \( (x_ix_j)^{m_{ij}} \) is in the kernel of \( \phi \). Since the kernel of \( \phi \) is a normal subgroup of \( F \), it follows that the kernel contains \( N \). Thus \( \phi \) factors as a homomorphism from \( W \) to \( G \), where \( \phi(s_i) = \phi(x_i N) = r_i \). It follows that the order of \( s_i s_j \) is at least the order of \( r_i r_j \), and since it is at most \( m_{ij} \) it must equal \( m_{ij} \). Hence (i) holds.

(ii) Suppose \( S' \) is any subset of \( S \), and \( i, 1 \leq i \leq n \) with \( s_i \in \langle S' \rangle \). Let \( V \) denote the subspace of \( V \) spanned by the vectors \( b_j, j \in I \), and let \( G_I \) be the subgroup of \( G \) generated by \( \{r_j \mid j \in I\} \). If \( s_i \in \langle S' \rangle \) then \( r_i \in G_I \) by the homomorphism \( \phi \) above (note that we are not assuming that \( i \in I \)). Now,

\[
  r_j(v) = v - 2 \langle v, b_j \rangle b_j, \quad b_j \in v + V_j \subset v + V_I,
\]

for all \( j \in I \), and hence \( r(v) \in v + V_I \) for all \( r \in G_I \). Hence,

\[
  r_i(v) = v - 2 \langle v, b_i \rangle b_i, \quad b_i \in v + V_I,
\]

and so \( r_i(v) \in v + V_I \) for all \( v \in V \). In particular, \(-b_i = r_i(b_i) \in b_i + V_I \), and thus \( b_i \in V_I \). But this implies that \( i \in I \), and hence \( s_i \in S' \). \( \square \)

**Corollary 2.8.** The map \( \langle S' \rangle \mapsto S' \) is a poset bijection.

Suppose \( W \) is a Coxeter group. Given any \( w \in W \), there is a set of elements \( s_1, \ldots, s_k \in S \) such that \( w \) equals the product \( w = s_1 \cdots s_k \), since \( S \) generates \( W \). There are many such products that equal any given \( w \in W \), and two different products that equal the same element in \( W \) does not necessarily have the same length. For example, if \( w = s_1 \cdots s_k \), then \( w = s_1 \cdots s_k s_1 s_1^{-1} \) as well, and this product has two more elements than the first. We say that the product \( s_1 \cdots s_k \) is a **minimal decomposition** of \( w \) if \( w = s_1 \cdots s_k \) and there is no product of elements of \( S \) of length less than \( k \) that equal \( w \).

The following lemma is very important, but the proof requires theory that would require too much time and space to fit in here, so it is given without proof. For a proof, see for example [1], Chapter II, Section 3C.

**Lemma 2.9.** For every element \( w \in W \) there is a unique set \( S(w) \subseteq S \) such that every reduced decomposition of \( w \) is a product of elements of \( S(w) \).

The subgroups of the form \( \langle S' \rangle \) where \( S' \subseteq S \) are, as we shall see in the next section, important enough to have their own name.

**Definition 2.10.** The subgroups of a Coxeter group on the form \( \langle S' \rangle \) where \( S' \subseteq S \) are called the **special subgroups** of \( W \), and the cosets of the special subgroups are called the **special cosets** of \( W \).

### 2.2 Coxeter complexes

All examples of Coxeter groups so far have been reflection groups. It can be shown that the finite Coxeter groups are precisely the finite reflection groups. Hence they have a lot of nice geometric properties. In fact, as we will see in this section, given a Coxeter group, one can construct a chamber complex with a labeling on which the Coxeter group acts type preserving and transitively on the set of chambers.

**Definition 2.11.** The **Coxeter complex** of a Coxeter group \( W \) is the set of special cosets of \( W \), ordered by opposite inclusion.

**Example 2.12** Let \( G = \{-1,1\} \) under multiplication. We have seen that it is a Coxeter group,
and its Coxeter complex is isomorphic to the simplicial complex containing two singletons

\[
\begin{array}{c}
\{1\} \\
\{1, -1\}
\end{array} \sim \begin{array}{c}
\{a\} \\
\emptyset
\end{array}
\]

We will shortly prove that this structure is in fact a thin chamber complex with rank equal to the cardinality of \( S \), but first we will investigate some of the properties of \( \Sigma(W, S) \). However, we will apply the terminology of chamber complexes freely, trusting that this will be justified later. For example, if \( w' \langle S' \rangle \leq w'' \langle S'' \rangle \) we say that \( w' \langle S' \rangle \) is a face of \( w'' \langle S'' \rangle \), the maximal elements are called chambers and so on.

Since \( \Sigma(W, S) \) is ordered by opposite inclusion, the order is a partial order. Also, the maximal elements (chambers) are the elements where \( S' = \emptyset \), i.e. the elements of the form \( w \langle \emptyset \rangle = \{w\} \). Hence the chambers of \( \Sigma(W, S) \) are simply the the singletons.

The elements on the form \( w \langle s \rangle = \{w, ws\} \ (s \in S) \) are said to be of co-dimension 1. Such an element is a face of the two chambers \( \{w\} \) and \( \{ws\} \). Two chambers sharing a co-dimension 1 face are said to be adjacent. Hence, the chamber \( \{w\} \) is adjacent to \( |S| + 1 \) chambers, which are precisely the chambers \( \{ws\}, s \in S \cup \{1\} \). Equivalently, two chambers \( \{w\} \) and \( \{w'\} \) are adjacent if and only if \( w = w' \) or \( w = w's \) for some \( s \in S \). Note that in particular, every chamber is adjacent to itself.

The vertices of \( \Sigma(S, W) \) are the elements whose only face is \( \langle S \rangle = W \). These are the special cosets on the form \( w \langle S \setminus \{s\} \rangle \), where \( s \in S \). Two vertices \( w \langle S \setminus \{s\} \rangle \) and \( w' \langle S \setminus \{s'\} \rangle \) are faces of a common chamber if and only if there is an element \( w'' \in W \) such that \( w'' \in w \langle S \setminus \{s\} \rangle \) and \( w'' \in w' \langle S \setminus \{s'\} \rangle \).

There is an obvious action of \( W \) on \( \Sigma(W, S) \), given by \( w' \langle w \langle S' \rangle \rangle = \langle w'w \rangle \langle S' \rangle \). This action is transitive on the chambers. The chamber \( \{1\} \) will be called the fundamental chamber and will be denoted by \( C \). Using the \( W \)-action, every chamber can be written (uniquely) as \( wC \), since

\[ \{w\} = w\{1\} = wC. \]

Furthermore, the action is order preserving, that is, if \( A, B \in \Sigma(W, S) \) and \( A \leq B \) then \( wA \leq wB \) for all \( w \in W \). Finally, the action preserves adjacency. As we have seen, two chambers \( wC, w'C \) are adjacent if and only if either \( w = w' \) or \( w = w's \) for some \( s \in S \). Hence when \( w'' \in W \) acts on these chambers, the result is \( w''wC \) and \( w''w'C \), and either \( w''w = w''w' \) or \( w''w = (w''w')s \), so adjacency is preserved.

**Example 2.13** Consider the Coxeter complex of the dihedral group of order 6, \( D_6 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^3 = 1 \rangle \). The chambers are (using the convention \( C = \{1\} \)):

\[
C, \ s_1C, \ s_2C, \ s_1s_2C, \ s_2s_1C, \ s_1s_2s_1C.
\]

Since \( \text{card}\{s_1, s_2\} = 2 \), the chambers are of rank 2. This Coxeter complex can be visualized as

and is hence the boundary of a 6-gon. In general, the Coxeter complex of \( D_{2n} \) is the boundary of a \( 2n \)-gon. The Coxeter complex of \( D_\infty \) is the boundary of an ‘infinite polygon’, i.e. the simplicial
complex

\[
\begin{array}{cccccc}
s_1 \langle s_2 \rangle & s_1 C & \langle s_1 \rangle & C & \langle s_2 \rangle & s_2 C & s_2 \langle s_1 \rangle \\
\end{array}
\]

We now prove that $\Sigma(W, S)$ in fact is a chamber complex.

**Proposition 2.14.** $\Sigma(W, S)$ is a thin chamber complex of rank equal to the cardinality of $S$ with a labeling. Furthermore, the action of $W$ on $\Sigma(W, S)$ is type preserving.

**Proof.** We start by showing that $\Sigma(W, S)$ is a simplicial complex. We need to show that (i) and (ii) of Proposition 1.13 hold.

(i) We will show that any two elements $w' \langle S' \rangle, w'' \langle S'' \rangle \in \Sigma(S, W)$ have a greatest lower bound. Since they are ordered by opposite inclusion, we must show that there is a least element $w''' \langle S''' \rangle$ such that $w' \langle S' \rangle \cup w'' \langle S'' \rangle \subseteq w''' \langle S''' \rangle$.

By using the $W$-action we can assume, without loss of generality, that $w'' = 1$. Now, as any special coset $w''' \langle S''' \rangle$ satisfying $w' \langle S' \rangle \cup \langle S'' \rangle \subseteq w''' \langle S''' \rangle$ includes the identity (since $1 \in \langle S'' \rangle$), we may assume that $w''' = 1$. Hence we need to find a least special subgroup $\langle S''' \rangle$ containing $w' \langle S' \rangle$ and $\langle S'' \rangle$. But by Lemma 2.9 the special subgroup $\langle S' \cup S'' \cup \langle w' \rangle \rangle$ is such a subgroup. It is obviously the least such subgroup, and so (i) follows.

(ii) Again, by the transitivity of the $W$-action on the chambers, it suffices to study a special subgroup $\langle S' \rangle$. Also, if (ii) holds for $\Sigma \leq C$ it clearly holds for $\Sigma \leq A$ for any face $A$ of $C$. But, as posets, we have

\[\Sigma \leq C \simeq \text{[subsets of } S\text{ under opposite inclusion]} \simeq \text{[subsets of } S\text{ under inclusion]},\]

where the first isomorphism follows from Corollary 2.8 and the second isomorphism is given by $S' \mapsto S \setminus S'$, and thus (ii) follows.

To show that $\Sigma(W, S)$ is a chamber complex we must show that all maximal simplexes have the same cardinality, and that there is a gallery connecting any two chambers. Clearly all chambers have the same rank, since they all have rank equal to the cardinality of $S$.

Now, using the $W$-action, we see that it suffices to find a gallery from $C$ to any chamber $wC$. Let $w = s_1 \cdots s_l$, and let $w_i = s_1 \cdots s_i, 1 \leq i \leq l$. Then $w_1 C = s_1 C$ is adjacent to $C$, $w_2 C = s_1 s_2 C$ is adjacent to $w_1 C$, and so on. Hence

\[C, w_1 C, w_2 C, \ldots, w_n C = wC\]

is a gallery from $C$ to $wC$. $\Sigma(W, S)$ is therefore a chamber complex. We have already established that any co-dimension 1 simplex is a face of exactly two chambers, and hence $\Sigma(W, S)$ is thin.

Finally, we define a labeling on $\Sigma(W, S)$ by $\lambda(w \langle S' \rangle) = S \setminus S'$. We need to see that this labeling is well defined, that is, if $w' \langle S' \rangle = w'' \langle S'' \rangle$, then $S' = S''$. If $w' \langle S' \rangle = w'' \langle S'' \rangle$ then $(S') = \langle w'^{-1}w'' \rangle \langle S'' \rangle$, and hence $1 \in \langle w'^{-1}w'' \rangle \langle S'' \rangle$ and hence $w'^{-1}w'' \in \langle S'' \rangle$, so that

\[\langle S' \rangle = w'^{-1}w'' \langle S'' \rangle = \langle S'' \rangle\]

and we are done. That the labeling is $W$-invariant is obvious. $\square$

The labeling $\lambda$ will be referred to as the **canonical labeling** of $\Sigma(W, S)$.

**Proposition 2.15.** The flag complex of the boundary of an $(n - 1)$-simplex is isomorphic to the Coxeter complex $\Sigma(W, S)$, where $W$ is the symmetric group on $n$ letters, and $S = \{s_i \mid 1 \leq i < n\}$, where $s_i$ is the transposition $(i \ i+1), 1 \leq i < n$.

**Proof.** Let the vertices of the $(n - 1)$-simplex be $1, 2, \ldots, n$, and denote the flag complex of this simplex by $\mathcal{N}$, then $W$ acts on $\mathcal{N}$ in the natural way. Let the fundamental chamber of $\mathcal{N}$ be

\[C = V_1 \subset \cdots \subset V_{n-1},\]

where $V_i = \{1, \ldots, i\}, 1 \leq i < n$. If

\[C' = V_1' \subset \cdots \subset V_{n-1}'\]
is any other chamber of \( \mathcal{N} \), let \( a_1 \) be the element in \( V'_i \) and let \( a_i \) be the element in \( V_i \setminus V_{i-1} \) for \( 1 < i < n \), and let \( a_n \) be the element in \( \{1, \ldots, n\} \setminus V_{n-1} \). Then \( w \in W \) defined by \( w(i) = a_i \) acts on \( C \), with \( wC = C' \) (see Figure 2.2). Hence \( C \) can be mapped to any chamber of \( \mathcal{N} \), and any chamber can be mapped to \( C \). Hence \( W \) is transitive on the chambers of \( \mathcal{N} \). Furthermore, the element \( w \) is uniquely determined by \( C' \), and hence a bijection from the chambers of \( \mathcal{N} \) to the chambers of \( \mathcal{N} \).

If \( A \) is any face of \( C \), then an element \( s_i \in S \) fixes \( A \) if and only if \( V_i \) is not in the flag \( A \). Hence, for any set \( S' \subseteq S \) there is a unique maximal face \( A(S'') \) fixed by \( \langle S' \rangle \), namely the flag containing all \( V_i, 1 < i < n \) except for those \( i \) where \( s_i \in S \) (the empty flag if \( S' = S \)). So there is a natural bijection \( S' \to A(S'') \) from subsets of \( S \) to faces of \( C \). This bijection is easily seen to be a poset isomorphism, i.e. \( S' \subseteq S'' \) if and only if \( A(S') \supseteq A(S'') \). Hence the simplex \( C \) is isomorphic to the simplex of special subgroups of \( W \).

Combining the results from the previous two paragraphs suggests a way to construct a poset isomorphism from \( \Sigma \) to \( \mathcal{N} \). Let \( A' \) be any simplex in \( \mathcal{N} \), and let \( C' \) be any chamber containing \( A' \). Let \( w \) be the element with \( wC = C' \), and let \( A = w^{-1}(A') \). Clearly, \( A \) is a face of \( C \) of the same rank as \( A' \), and \( A' = w(A) \). If \( A' \) is a vertex, i.e. the flag contains only one set, then this set is mapped to a set with the same cardinality, as \( w \) is a permutation. Hence, if \( C' \) is a maximal simplex containing the vertex \( A' \), and \( w' \) and \( w'' \) are the two elements mapping \( C \) to \( C' \) and \( C'' \) respectively, then \( w'^{-1}(A') = w''^{-1}(A') \). As this is true for any vertex of \( \mathcal{N} \), it is also true for any simplex of \( \mathcal{N} \). In addition, there is a unique maximal set \( S' \subseteq S \) with \( A = A(S') \). Hence \( A' = wA(S') \).

Now define \( \phi : \Sigma \to \mathcal{N} \) by \( \phi(w(S')) = w(A(S')) \). This function is obviously surjective. To see that it is injective, suppose that for some \( w', w'' \in W \) and some \( S' \subseteq S \) we have that \( w'A(S') = w''A(S') \). Then

\[ A(S') = w'^{-1}w''A(S') \]

and hence \( w'^{-1}w'' \) fixes \( A(S') \). Thus \( w'^{-1}w'' \in \langle S' \rangle \), and hence \( w'\langle S' \rangle = w''\langle S' \rangle \), and so \( \phi \) is injective.

Finally, to see that \( \phi \) is a poset isomorphism, it suffices to note that \( S'' \subseteq S' \subseteq S \) obviously implies \( A(S'') \subseteq A(S') \), and that the action of \( w \) is a poset automorphism of \( \mathcal{N} \).

We will now turn our attention to the Coxeter group defined in Example 2.6, \( \mathbb{Z}^n \rtimes S_{n+1} \). Before we investigate the Coxeter complex of this group, we will look at some of the properties of its special subgroups on the form \( w \langle S \setminus \{s_i\} \rangle \), that is, the vertices of its Coxeter complex.

**Proposition 2.16.** Let \( W = \mathbb{Z}^n \rtimes S_{n+1} \) and let \( S = \{s_0, \ldots, s_n\} \) be the set of generators as defined in Example 2.6. This group has the following properties:

(i) Every special subgroup on the form \( \langle S \setminus \{s_i\} \rangle, 0 \leq i \leq n \) is isomorphic to \( S_{n+1} \).

(ii) For all \( s \in S_{n+1} \) and \( s_i \in S \) there is an element \( w \in \langle S \setminus \{s_i\} \rangle \) such that \( w = [x, s] \) for some \( x \in \mathbb{Z}^n \).

(iii) The map \( \phi : \langle S \setminus \{s_i\} \rangle \to S_{n+1} \) defined by \( \phi_i([x, s]) = s \) is an isomorphism for all \( 0 \leq i \leq n \).

(iv) The element \([x, 1]\) is in the special subgroup \( \langle S \setminus \{s_i\} \rangle \) if and only if \( x = 0 \).
(v) For any special coset of $W$ on the form $w \langle S \setminus \{ s_i \} \rangle$ there is a unique element $x \in \mathbb{Z}^n$ with $w \langle S \setminus \{ s_i \} \rangle = [x, 1] \langle S \setminus \{ s_i \} \rangle$.

Proof.

(i) All groups on the form $\langle S \setminus \{ s_i \} \rangle$, $0 \leq i \leq n$ are isomorphic since they all admit the same presentation. Also, the special subgroup $\langle S \setminus \{ s_0 \} \rangle$ is seen to be isomorphic to $S_{n+1}$, since $\langle S \setminus \{ s_0 \} \rangle$ is simply the set 

$$\{ [0, \sigma] \mid \sigma \in S_{n+1} \}.$$ 

(ii) The ‘$S_{n+1}$’ part of the elements of $S$ are the transpositions $(12), \ldots, (n \ n+1), (1 \ n+1)$. If we remove just one of these, we still have enough transpositions to generate $S_{n+1}$.

(iii) Since

$$\phi_i([x, \sigma] [y, \tau]) = \phi_i([x + \sigma(y), \sigma \tau]) = \sigma \tau = \phi_i([x, \sigma]) \phi_i([y, \tau]),$$

the map $\phi_i$ is a homomorphism. (i) implies that $\langle S \setminus \{ s_i \} \rangle$ is finite, and (ii) shows that $\phi_i$ is onto, and hence it is a bijection.

(iv) This follows from the fact that $\phi_i$ is an isomorphism, since $\phi_i^{-1}(1)$ must equal the identity in $\mathbb{Z}^n \rtimes S_{n+1}$, which is $[0, 1]$. Hence $[0, 1]$ is the only element of $\langle S \setminus \{ s_i \} \rangle$ such that the ‘$S_{n+1}$’ part is the identity.

(v) Let $w = [y, \sigma]$. Then, by (ii), there is a unique element $w' \in \langle S \setminus \{ s_i \} \rangle$ with $w' = [z, \sigma^{-1}]$. So, by putting $x = y + \sigma(z)$, we have

$$w \langle S \setminus \{ s_i \} \rangle = w w' \langle S \setminus \{ s_i \} \rangle = [y, \sigma] [z, \sigma^{-1}] \langle S \setminus \{ s_i \} \rangle = [y + \sigma(z), 1] \langle S \setminus \{ s_i \} \rangle = [x, 1] \langle S \setminus \{ s_i \} \rangle,$$

as required.

\[\square\]

Example 2.17 We will now show that the Coxeter complex of the group $W = \mathbb{Z}^n \rtimes S_{n+1}$ is isomorphic to the simplicial complex defined in Example 1.19.

Let $B = \{ b_1, \ldots, b_n \}$ be the basis of $\mathbb{R}^n$ defined in Example 2.6, and let $W$ act on $\mathbb{R}^n$ in the natural way. That is, if $[x, \sigma] \in W$, $y \in \mathbb{R}^n$, then

$$[x, \sigma] y = x + \sigma(y),$$

where the action of $S_{n+1}$ on $\mathbb{R}^n$ is as defined in Example 2.6. This is indeed a group action, since

$$[0, 1] x = 0 + 1(x) = x$$

and

$$([x, \sigma][y, \tau]) z = [x + \sigma(y), \sigma \tau] z = x + \sigma(y) + \sigma \tau(z)
= x + \sigma(y + \tau(z)) = [x, \sigma][y, \tau] z.$$ 

The action of $[x, \sigma]$ on $y$ is quite simple; first $\sigma$ rotates/reflects the vector $y$ about the origin, and then the point $\sigma(y)$ is translated by the integer vector $x$.

We will now investigate the fixed points of the special subgroups $\langle S \setminus \{ s_i \} \rangle$. For this, we need to know how all $s_i, 0 \leq i \leq n$ acts. We already know the action of $s_1, \ldots, s_n$ since these act as $s_1, \ldots, s_n$ as investigated in Example 2.6. Thus, we only need to understand the action of $s_0$. Using a similar same technique as in Example 2.6, we find that $s_0$ acts by reflection in the affine hyperplane $2x_1 + x_2 + \cdots + x_n = 1$. Figure 2.3 shows how these generators act in case $n = 2$.

It is obvious that the only fixed point of $\langle S \setminus \{ s_0 \} \rangle$ is the origin. The fixed points of $\langle S \setminus \{ s_1 \} \rangle$ are a bit more complicated to work out (but not much). These are the points that are fixed by $s_0, s_2, \ldots, s_n$, and are thus the intersection of the hyperplanes that define these reflections. These are $x_2 = x_3, \ldots, x_{n-1} = x_n, x_1 + \cdots + x_{n-1} + 2x_n = 0$ and $2x_1 + x_2 + \cdots + x_n = -1$. The first of these equations imply that $x_2 = \cdots = x_n$, which reduces the last two equations to

$$x_1 + nx_n = 0, \quad 2x_1 + (n-1)x_n = -1,$$
Thus we need to show that \( \langle \text{point of } \Sigma(\mathbb{W}, S) \rangle \) is well-defined. Assume \([x, \sigma] \in \Sigma(\mathbb{W}, S)\), where \([x, \sigma], [y, \tau] \in \mathbb{R}^n \times S_{n+1}, 0 \leq i \leq n\). Then there is an element \([z, \rho] \in \Sigma(\mathbb{W}, S)\) with \([x, \sigma] = [y, \tau][z, \rho]\). But then (since \(\sigma, \tau\) and \(\rho\) are linear and \([z, \rho]\) fixes \(p_i\))

\[
\phi([x, \sigma] \langle \Sigma(\mathbb{W}, S) \rangle) = \phi([y, \tau][z, \rho] \langle \Sigma(\mathbb{W}, S) \rangle) = \phi([y + \tau(z), \tau\rho(\mathbb{W}, S) \langle \Sigma(\mathbb{W}, S) \rangle])
\]

\[
= y + \tau(z) + \tau\rho(p_i) = y + \tau(z + \rho(p_i)) = y + \tau([z, \rho]p_i)
\]

\[
= y + \tau(p_i) = \phi([y, \tau] \langle \Sigma(\mathbb{W}, S) \rangle).
\]

Thus \(\phi\) is well-defined.

\(\phi\) is obviously surjective. To see that \(\phi\) is injective, suppose that \(\phi(w') \langle \Sigma(\mathbb{W}, S) \rangle = \phi(w' \mathbb{W}, S) \langle \Sigma(\mathbb{W}, S) \rangle\) for some \(w, w' \in \mathbb{W} \times S_{n+1}, 0 \leq i \leq n\). Then \(w'p_i = wp_i\) and hence \(w^{-1}w'p_i = p_i\). Thus \(w^{-1}w' \in \Sigma(\mathbb{W}, S)\) which implies \(w' \langle \Sigma(\mathbb{W}, S) \rangle = w \langle \Sigma(\mathbb{W}, S) \rangle\).

Now define a relation on \(\Xi\) by saying that \(wp_i\) is related to \(w'p_j\) if and only if the vertices \(\phi^{-1}(wp_i) = w \langle \Sigma(\mathbb{W}, S) \rangle\) and \(\phi^{-1}(w'p_j) = w' \langle \Sigma(\mathbb{W}, S) \rangle\) of \(\Sigma(W, S)\) are vertices of a common chamber. Note that this implies that two points of \(\Xi\) are related if and only if there is an element \(w \in W\) and indices \(i\) and \(j\) such that the two points are \(wp_i\) and \(wp_j\). The flag complex of \(\Xi\) is then isomorphic to the complex \(\Sigma(W, S)\). Note that, by taking the convex hull of related points
of \( \Xi \) we get a geometric realization of \( \Sigma(W,S) \) in \( \mathbb{R}^n \). \( \phi \) is naturally extended to an isomorphism from \( \Sigma(W,S) \) to the flag complex of \( \Xi \) by putting

\[
\phi(w \langle S \setminus S' \rangle) = \bigcup_{i \mid \tilde{s}_i \in S'} \{ wp_i \}.
\]

We label the elements of the flag complex of \( \Xi \) by putting

\[
\lambda(\cup_{i \in I} \{ wp_i \}) = I,
\]

\( I \subseteq \{0, \ldots, n\} \), which is easily seen to be a labeling. Figure 2.4 illustrates the points of \( \Xi \) in case \( n = 2 \).

To see that this chamber complex is isomorphic to the one defined in Example 1.19 we introduce the basis \( A = \{a_1, \ldots, a_n\} \) of \( \mathbb{R}^n \), where

\[
a_1 = \left(\frac{-n}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)
\]
\[
a_2 = \left(\frac{1}{n+1}, \frac{-n}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)
\]
\[
\vdots
\]
\[
a_n = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}, \frac{-n}{n+1}\right).
\]

This is like a natural basis for the elements of \( \Xi \). For example, the points \( p_i, 1 \leq i \leq n \) can be written as

\[
p_i = \sum_{j=1}^{i} a_j.
\]

The action of \( S_{n+1} \) on \( \mathbb{R}^n \) can be written quite easily in terms of the basis \( a_1, \ldots, a_n \). For \( \sigma \in S_{n+1} \) and \( 1 \leq i \leq n \) it is given by

\[
\sigma(a_i) = \begin{cases} 
\sigma(i) & \text{if } \sigma(i) \neq n+1 \\
- \sum_{j=1}^{n} a_j & \text{if } \sigma(i) = n+1
\end{cases}
\]

The vectors of the basis \( b_1, \ldots, b_n \) can also be easily written in terms of the basis \( a_1, \ldots, a_n \). They are

\[
b_i = -a_i - \sum_{j=1}^{n} a_j, 1 \leq i \leq n.
\]

Notice that this implies that all vertices in \( \Xi \) have integer coordinates with respect to the basis \( A \). It follows from Equation (2.3) that the matrix expressing the change of basis from \( B \) to \( A \) is

\[
T = -I - J,
\]

where \( I \) is the identity matrix and \( J \) is the \( n \times n \)-matrix with all entries equal to 1.
2.2. COXETER COMPLEXES

As we have previously seen, two vertices of Ξ are related if and only if they can be written as \( wp_i \) and \( wp_j \) for some \( w = [x, \sigma] \in W \) and \( 1 \leq i, j \leq n \). But, if \( i < j \),

\[
w p_i - wp_j = x + \sigma(p_i) - x - \sigma(p_j) = \sigma(p_i - p_j)
\]

and if \( i > j \)

\[
w p_i - wp_j = x + \sigma(p_i) - x - \sigma(p_j) = \sigma(p_i - p_j)
\]

In any case, if we write

\[
w p_i - wp_j = \sum_{k=1}^{n} q_k a_k,
\]

then either \( q_k \in \{0, 1\} \) for all \( k \) or \( q_k \in \{-1, 0\} \) for all \( k \). Considering all the possibilities for \( i \) and \( j \), we see that two points \( wp_i \) and \( wp_j \) of Ξ are related if and only if

\[
w p_i - w'p_j = \sum_{k=1}^{n} q_k a_k,
\]

where either \( q_k \in \{0, 1\} \) for all \( 1 \leq k \leq n \) or \( q_k \in \{-1, 0\} \) for all \( 1 \leq k \leq n \). But this is precisely the relation on \( \mathbb{Z}^n \) defined in Example 1.19, and hence we have found a bijection \( \theta \) from the vertices of \( \Sigma(W, S) \) to \( \mathbb{Z}^n \) such that \( x \in \mathbb{Z}^n \) is related to \( y \in \mathbb{Z}^n \) by the relation defined in Example 1.19 if and only if \( \theta^{-1}(x) \) and \( \theta^{-1}(y) \) are vertices of a common chamber, where \( \theta \) is given by

\[
\theta([x, \sigma]) = T(x + \sigma(p_i)).
\]

Also, \( \theta \) naturally extends to a simplicial isomorphism from \( \Sigma(W, S) \) to the simplicial complex defined in Example 1.19.

We will end this example by looking at what the geometric realization of \( \Sigma(W, S) \) looks like in the hyperplane \( H \) of \( \mathbb{R}^n \) under the bijection \( \psi \), as defined in Example 2.6. We will need this description in Chapter 5.

We can identify the action of \( \mathbb{Z}^n \rtimes S_{n+1} \) of \( H \) with the natural action of \( (H \cap \mathbb{Z}^{n+1}) \rtimes S_{n+1} \), where \( (H \cap \mathbb{Z}^{n+1}) \) acts by translation and \( S_{n+1} \) acts by permutation of coordinates. The action of \( \tilde{s}_1 \) on \( H \) is given by

\[
\tilde{s}_1(t_1, \ldots, t_{n+1}) = (t_2, t_1, t_3, \ldots, t_{n+1}).
\]

This is a linear transformation which has a 1-eigenspace given by the hyperplane \( t_1 = t_2 \), and a \((-1)\)-eigenspace given by the line \( t_1 = t_2, t_3 = \cdots = t_{n+1} = 0 \), as we have seen before. A basis for the 1-eigenspace is

\[
\{b_1 + b_2 - 2b_{n+1}, b_3 - b_{n+1}, \ldots, b_n - b_{n+1}\}
\]

where \( \{b_1, \ldots, b_{n+1}\} \) is the standard basis of \( \mathbb{R}^{n+1} \). A basis for the \((-1)\)-eigenspace is \( \{b_1 - b_2\} \). It is easy to see that the \((-1)\)-eigenspace is orthogonal to the 1-eigenspace, and hence \( \tilde{s}_1 \) acts on \( H \) by reflection in the hyperplane \( t_1 = t_2 \). Similarly, \( \tilde{s}_i \) acts on \( H \) as a reflection in the hyperplane \( t_i = t_{i+1} \) for \( i, 1 \leq i \leq n \), and \( \tilde{s}_i \) acts on \( H \) by reflection in the hyperplane \( t_1 = 1 = t_{n+1} \). Note however, that \( \tilde{s}_i \) does not act as a reflection on \( \mathbb{R}^{n+1} \), but only when restricted to \( H \).
Now, the vertices of the fundamental chamber in \( H \) are \( \psi^{-1}(p_i), 0 \leq i \leq n \). That is,

\[
\begin{align*}
\psi^{-1}(p_0) &= (0, \ldots, 0) \\
\psi^{-1}(p_1) &= \left( \frac{-n}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1} \right) \\
& \quad \vdots \\
\psi^{-1}(p_i) &= \left( \frac{i-n-1}{n+1}, \frac{i-n-1}{n+1}, \ldots, \frac{i}{n+1}, \ldots, \frac{i}{n+1} \right) \\
& \quad \vdots \\
\psi^{-1}(p_n) &= \left( -\frac{1}{n+1}, \ldots, -\frac{1}{n+1} \right).
\end{align*}
\]

The convex hull of these points is the intersection of the half-spaces \( t_1 \leq t_2, \ldots, t_n \leq t_{n+1} \leq t_1 + 1 \). That is, the set

\[
C = \{(t_1, \ldots, t_{n+1}) \in H \mid t_1 \leq \cdots \leq t_{n+1} \leq t_1 + 1 \}.
\]

The geometric realization in \( H \) of any chamber of \( \Sigma(W, S) \) is the set \( wC \), where \( w \in \mathbb{Z}_n \rtimes S_{n+1} \) is the element representing the chamber.

As before, \( [0, \sigma] \) acts on \( H \) as a reflection if and only if \( \sigma \) is a transposition. Also, given the transposition \( \tau_{ij}, 1 \leq i, j \leq n \), if the element \([(x_1, \ldots, x_{n+1}), \tau_{ij}]\) is a reflection then by necessity \( x_l = 0 \) for all \( l \neq i, j \), as otherwise it would not fix any points. Also, for this element we have \( t_i \mapsto t_j + x_i \) and \( t_j \mapsto t_i + x_j \). Hence for it to fix any points we must have \( x_i = -x_j \). We conclude that the reflections of \( \mathbb{Z}_n \rtimes S_{n+1} \) are precisely the reflections in the planes

\[
H(i, j; k) = \{(t_1, \ldots, t_{n+1}) \mid t_i - t_j = k \}
\]

of \( H \), for \( 1 \leq i, j \leq n, k \in \mathbb{Z} \).
Chapter 3

Buildings

The theory of buildings was developed for finding geometries associated to certain groups. This area is vast, and this chapter only gives a brief introduction to this theory. For a more extensive introduction, see for example [1] and [10].

3.1 Buildings

Definition 3.1. A building is a simplicial complex $\Delta$ with a set of subcomplexes called apartments satisfying

(B0) every apartment is a Coxeter complex,

(B1) for any two simplices $A, B \in \Delta$, there is an apartment $\Sigma$ containing both of them,

(B2) If $\Sigma$ and $\Sigma'$ are two apartments with a common chamber, there is an isomorphism $\Sigma \rightarrow \Sigma'$ fixing $\Sigma \cap \Sigma'$ pointwise.

Note that (B1) and (B2) imply that any two apartments are isomorphic.

Example 3.2 Let $\Pi$ be the projective plane of order two (see Figure 3.1a), let $V$ be the set of points and lines of $\Pi$, and let $\Delta$ be the flag complex of $V$ where the relation is given by point/line incidence. This is a building, where an apartment is given by any set of three lines with no point in common, and their three points of intersection (or conversely, any set of three non-collinear points together with the three lines they determine). This building is shown in Figure 3.1b.

That this indeed is a building will follow from the next example, but note that, for example, each apartment is a 6-gon, which is the Coxeter complex of the group $S_2$.

Example 3.3 Let $V$ be a vector space of finite dimension $n \geq 2$ over an arbitrary field and let

![Figure 3.1: The projective plane of order two (a) and its flag complex (b).](image)
Δ be the flag complex of proper subspaces of V (which are ordered by inclusion). A \textit{frame} in V is a set \( F = \{ L_1, \ldots, L_n \} \) of 1-dimensional subspaces of V such that \( V = L_1 \oplus \cdots \oplus L_n \). We will show that this flag complex is a building, where the apartments are the flag complexes of a given frame. We will do this in three steps, showing that

(i) the flag complex of subspaces spanned by non-empty proper subsets of \( F \) is a Coxeter complex, so that \( \Delta \) satisfies (B0),

(ii) that any two maximal flags in \( \Delta \) have a frame that contains both of them, thus so that \( \Delta \) satisfies (B1)

(iii) for any to frames with a maximal flag in common there is an isomorphism between them fixing every flag common to both of them, so \( \Delta \) satisfies (B2).

First of all, we define a labeling on \( \Delta \) by setting \( \lambda(A) \) to be the set of dimensions that occur in the flag \( A \). For example, if \( A = V_1 \subset V_2 \subset V_3 \), where \( V_1 \) has dimension 2, \( V_2 \) dimension 3 and \( V_3 \) dimension 5, then \( \lambda(A) = \{2, 3, 5\} \).

(i) It suffices to note that the map that takes any subspace \( V' = \bigoplus_{i \in I'} L_i, I' \subset I \) to the set \( \{i \mid i \in I'\} \) is a poset isomorphism from the flag complex of \( F \) to the flag complex of the boundary of an \((n-1)\)-simplex, which by Proposition 2.15 is a Coxeter complex.

(ii) Let the two maximal simplices be

\[
V_1 \subset \cdots \subset V_n
\]

and

\[
V'_1 \subset \cdots \subset V'_n.
\]

We will define a permutation \( \pi \) on the set \( \{1, \ldots, n\} \) with \( V_i = \bigoplus_{j=1}^i L_j \) and \( V'_i = \bigoplus_{j=1}^i L_{\pi(j)} \)

for some 1-dimensional subspaces \( L_i, 1 \leq i \leq n \).

Let \( j = \pi(i) \) be the smallest integer such that

\[
V'_{i-1} + V'_i \cap V_j = V'_i.
\]

There is certainly some such \( j \), since \( j = n \) satisfies the equality and \( j > 0 \). We will show that \( \pi \) is indeed a permutation by showing that if \( j = \pi(i) \), then \( i \) is the smallest integer with

\[
V_{j-1} + V_j \cap V'_i = V_j.
\]

We shall show two things; that \( i \) indeed satisfies Equation 3.1, and that it is the smallest such integer. Let \( j = \pi(i) \). Clearly

\[
V_{j-1} \subseteq V_{j-1} + V_j \cap V'_i \subseteq V_j,
\]

and one of these inequalities is a strict inequality and the other is an equality.

If \( V_{j-1} + V_j \cap V'_i = V_j \) we must have \( V_j \cap V'_i \nsubseteq V_{j-1} \). So assume, for a contradiction, that \( V_j \cap V'_i \nsubseteq V_{j-1} \). But then

\[
V'_{i-1} + V'_i \cap V_{j-1} \supseteq V'_{i-1} + V'_i \cap (V_j \cap V'_i) = V'_{i-1} + V'_i \cap V_j = V'_i,
\]

which contradicts the fact that \( j \) is the smallest such integer. Hence \( V_{j-1} + V_j \cap V'_i = V_j \), so \( i \) satisfies Equation 3.1.

Now, we have \( V_j \cap V'_{i-1} \subset V_{j-1} \). Since otherwise,

\[
V_j = V_{j-1} + V_j \cap V'_{i-1}.
\]

Intersecting both sides of Equation (3.2) with \( V'_i \), we get

\[
V'_i \cap V_j = V'_i \cap V_{j-1} + V_j \cap V'_{i-1} \subset V'_{i-1},
\]

which contradicts the definition of \( j \), and hence \( V_j \cap V'_{i-1} \nsubseteq V_{j-1} \). Now, suppose that there exists some \( k < i \) that satisfies Equation 3.1. Then \( V_j \cap V'_k \nsubseteq V_{j-1} \), but \( V_j \cap V'_k \subseteq V_j \cap V'_{i-1} \subseteq V_{j-1} \), so no such \( k \) exists.
Hence \( \pi \) is a bijection, i.e. a permutation of \( \{1, \ldots, n\} \), and \( \pi(i) \) is the smallest integer with
\[
V'_{i-1} + V_i' \cap V_{\pi(i)} = V_i'
\]
and \( \pi^{-1}(i) \) is the smallest integer with
\[
V_{i-1} + V_i \cap V_{\pi^{-1}(i)} = V_i.
\]

Now, let \( L_i \) be a 1-dimensional subspace of \( V_i' \cap V_{\pi(i)} \) that is not in \( V'_{i-1} \). By the construction of \( \pi \) it is clear that \( V_i = \bigoplus_{k=1}^i L_k \), and \( V'_i = \bigoplus_{k=1}^i L_{\pi(k)} \).

iii) Let the two flags be \( F = \{L_1, \ldots, L_n\} \) and \( F' = \{L'_1, \ldots, L'_n\} \), and the maximal flag be \( V_1 \subset \cdots \subset V_n \). Assume, without loss of generality, that the 1-dimensional subspaces \( L_i \) and \( L'_i \) are numbered in such a way that \( V_i = \bigoplus_{k=1}^i L_k = \bigoplus_{k=1}^i L'_k \). Then the mapping \( L_i \mapsto L'_i \) obviously maps the apartment associated to \( F \) to the apartment associated to \( F' \), fixing every common simplex.

It can be shown that the geometric realization of this building is homeomorphic to the sphere \( S^{n-1} \), and for this reason this building is said to be spherical.

**Example 3.4** Before reading this example, it is advisable to read through the appendix, as we will refer to theory, methods, results and notations described there.

Let \( F \) be a field with a discrete valuation \( \omega \) and valuation ring \( \mathcal{O} \) and let \( \pi \in \mathcal{O} \) be an element such that \( \omega(\pi) = -1 \).

Consider the set of lattice classes of \( F^n \). Since \( \text{GL}_n(F) \) acts transitively on the set of lattice classes, every lattice class \( \Lambda \) can be written as \( g \Lambda_0 \) for some \( g \in \text{GL}_n(F) \). Suppose that we have two elements \( g, g' \in \text{GL}_n(F) \) with \( \Lambda = g \Lambda_0 = g' \Lambda_0 \). Then \( g^{-1}g' \Lambda_0 = \Lambda_0 \), so \( g^{-1}g' \in Z \cdot \text{GL}_n(\mathcal{O}) \).

Let \( z = \text{diag}(\delta_1, \ldots, \delta) \in Z, h \in \text{GL}_n(\mathcal{O}) \). Then
\[
\omega \circ \det(g') = \omega \circ \det(gzh) = \omega \circ \det(g) + \omega \circ \det(z) = \omega \circ \det(g) + \omega(\delta),
\]
where the last equality follows since \( \det(z) = \delta^n \) and \( \det(h) \in \mathcal{O}^* \) for all \( h \in \text{GL}_n(\mathcal{O}) \). Hence \( \omega \circ \det(g) \equiv \omega \circ \det(g') \pmod{n} \) for all \( g, g' \in \text{GL}_n(F) \) with \( g \Lambda_0 = g' \Lambda_0 \). We can thus define a labeling \( \lambda \) on the set of lattice classes, by letting \( \lambda(g \Lambda_0) = \omega \circ \det(g) \pmod{n} \). In particular, \( \lambda(\Lambda_0) = 0 \). Also, if \( \Lambda = g \Lambda_0 \) is any lattice class, and \( g' \in \text{GL}_n(F) \), then
\[
\lambda(g' \Lambda) \equiv \lambda(g'g \Lambda_0) = \omega \circ \det(g'g) \equiv \omega \circ \det(g') + \omega \circ \det(g) \equiv \omega \circ \det(g') + \lambda(\Lambda) \pmod{n}.
\]

Define an incidence relation on the set of lattice classes by saying that the classes \( \Lambda \) and \( \Lambda' \) are incident (written \( \Lambda \sim \Lambda' \)) if there are representatives \( L \in \Lambda, L' \in \Lambda' \) with
\[
L \supset L' \supset \pi L.
\]
(3.3)

Note that the incidence relation is symmetric, since if \( L \supset L' \supset \pi L \) for two representatives \( L \) and \( L' \), then the representatives \( \pi L, L' \) satisfy \( L' \supset \pi L \supset \pi L' \). For example, if \( b_1, \ldots, b_n \) is any basis, then \([b_1, \ldots, b_n]\) is incident with \([\pi b_1, \ldots, \pi b_n]\), since
\[
[b_1, \ldots, b_n] \supset [b_1, \ldots, b_{n-1}, \pi b_n] \supset [\pi b_1, \ldots, \pi b_n].
\]

If \( \Lambda_1, \ldots, \Lambda_k \) are \( k \) pairwise incident lattice classes, then there are representatives \( L_1 \in \Lambda_1, \ldots, L_k \in \Lambda_k \) with
\[
L_1 \supset \cdots \supset L_k \supset \pi L_1.
\]

Consider the quotient \( L_1/\pi L_1 \), which is an \( n \)-dimensional \( \kappa \)-vector space. The quotients \( L_i/\pi L_1 \) are subspaces of this vector space, and \( L_i/\pi L_1 \) is a proper subspace of \( L_j/\pi L_1 \) if \( i < j \). It follows that every maximal set of incident lattice classes is of cardinality \( n \), since every chain of subspaces
of \(L_1/\pi L_1\) can be extended to a maximal chain that has length \(n\). If we have a maximal set of pairwise incident lattice classes, with representatives \(L_1, \ldots, L_n\) satisfying

\[
L_1 \supset \cdots \supset L_n \supset \pi L_1,
\]

then we can construct a basis \(\tilde{B} = \{\tilde{b}_1, \ldots, \tilde{b}_n\}\) of \(L_1/\pi L_1\) by letting \(\tilde{b}_n\) be a vector spanning the one-dimensional subspace \(L_n/\pi L_1\), \(\tilde{b}_{n-1}\) be a vector such that \(\{\tilde{b}_n, \tilde{b}_{n-1}\}\) spans \(L_{n-1}/\pi L_1\) and so on. Now, let \(b_i \in L_1\) be a pre-image of \(\tilde{b}_i\). Then \(B = \{b_1, \ldots, b_n\}\) is a basis of \(L_1\) with

\[
L_1 = [b_1, \ldots, b_n],
L_2 = [\pi b_1, b_2, \ldots, b_n]
\]

\[
\vdots
\]

\[
L_n = [\pi b_1, \ldots, \pi b_{n-1}, b_n].
\]

It follows that two incident lattice classes are of different type, since we can write \(L_i = d_i L_1\), where \(d_i\) is the element of \(GL_n(F)\) whose matrix representation with respect to the base \(B\) is the diagonal matrix \(\text{diag}(a_1, \ldots, a_n)\) with entries \(a_{ij} = \pi\) if \(j < i\) and \(a_{ij} = 1\) otherwise. Hence

\[
\lambda(L_i) = \lambda(d_i L_1) \equiv \omega \circ \det(d_i) + \lambda(L_1) \equiv \omega(\pi^{i-1}) + \lambda(L_1)
\]

\[
\equiv i - 1 + \lambda(L_1) \pmod{n},
\]

and we conclude that incident classes have different labelings.

Now consider the flag complex \(\Delta\) of these lattice classes. The maximal simplices all have rank \(n\), i.e. they are chambers. Let the apartments of \(\Delta\) be the subcomplexes obtained when restricting the lattice classes to a given basis. Thus, given the basis \(b_1, \ldots, b_n\), the apartment of this basis is the subcomplex given by the lattice classes on the form \([\pi^{i_1} b_1, \ldots, \pi^{i_n} b_n], i_1, \ldots, i_n \in \mathbb{Z}\). The apartment corresponding to the basis \(B\) is denoted \(\Sigma_B\). We have already seen that for any chamber \(C\) there is an apartment containing \(C\). We will now show that this is a building by showing that it satisfies the properties (B0), (B1) and (B2).

(B0) Let the given basis for the apartment be \(\{b_1, \ldots, b_n\}\), and denote it by \(\Sigma\). We start by noting that given any lattice class in this apartment, by scaling with an appropriate element of \(F^*\) we may assume that the coefficient of \(b_n\) is one. Also, as we saw above we may assume that the other coefficients are all powers of \(\pi\). This means that for any lattice class in this apartment there are unique integers \(l_1, \ldots, l_{n-1}\) such that the lattice class is

\[
[\pi^{l_1} b_1, \ldots, \pi^{l_{n-1}} b_{n-1}, b_n].
\]

Define a bijection \(\phi\) from \(\Sigma\) to \(\mathbb{Z}^{n-1}\) by

\[
\phi([\pi^{l_1} b_1, \ldots, \pi^{l_{n-1}} b_{n-1}, b_n]) = (l_1, \ldots, l_{n-1}).
\]

Now, two lattice classes \(\Lambda = [\pi^{l_1} b_1, \ldots, \pi^{l_{n-1}} b_{n-1}, b_n]\) and \(\Lambda' = [\pi^{l_1'} b_1, \ldots, \pi^{l_{n-1}'} b_{n-1}, b_n]\) of \(\Sigma\) are incident if and only if either \(l_i - l_i' \in \{0, 1\}\) for all \(1 \leq i \leq n - 1\) or \(l_i - l_i' \in \{0, -1\}\) for all \(1 \leq i \leq n - 1\). Thus \(\Lambda\) is incident with \(\Lambda'\) if and only if \(\phi(\Lambda)\) is related to \(\phi(\Lambda')\), where the relation on \(\mathbb{Z}^{n-1}\) is the one defined in Example 1.19. Hence \(\Sigma\) is isomorphic to the simplicial complex of Example 1.19, which in turn is the Coxeter complex of the group \(\mathbb{Z}^{n-1} \rtimes S_n\). Thus, \(\Delta\) satisfies (B0).

(B1) Let the two chambers be \(C\) and \(C'\). By the argument above, there are bases \(B = \{b_1, \ldots, b_n\}\) and \(B' = \{b_1', \ldots, b_n'\}\) with

\[
C = [L_1] \sim \cdots \sim [L_n]
\]

and

\[
C' = [L'_1] \sim \cdots \sim [L'_n],
\]

where \(L_i = [\pi b_1, \ldots, \pi b_{i-1}, b_i, \ldots, b_n]\) and \(L'_i = [\pi b_1', \ldots, \pi b_{i-1}', b_i, \ldots, b_n']\). The stabilizer of \(L_i\) is the group \(d_i \mathrm{GL}_n(O_B) d_i^{-1}\), where \(d_i\) is the diagonal element described above that
maps $L_1$ to $L_i$. Written as matrices with respect to the basis $B$, this group is the group of matrices of the form

$$d_i \text{GL}_n(\mathcal{O})_B d_i^{-1} = \begin{bmatrix}
\pi & \cdots & 0 \\
\cdots & \pi & \cdots \\
0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\mathcal{O} & \cdots & \mathcal{O} \\
\cdots & \mathcal{O} & \cdots \\
\mathcal{O} & \cdots & \mathcal{O}
\end{bmatrix}
\begin{bmatrix}
\pi^{-1} & \cdots & 0 \\
\cdots & \pi^{-1} & \cdots \\
0 & \cdots & 1
\end{bmatrix}$$

Hence, the stabilizer of $C$ is the group $\bigcap_{i=1}^n d_i \text{GL}_n(\mathcal{O})_B d_i^{-1}$. This is the group whose matrices with respect to the basis $B$ is of the form

$$\bigcap_{i=1}^n d_i \text{GL}_n(\mathcal{O})_B d_i^{-1} = \begin{bmatrix}
\mathcal{O} & \cdots & \mathcal{O} \\
\cdots & \mathcal{O} & \cdots \\
\mathcal{O} & \cdots & \mathcal{O}
\end{bmatrix}
\begin{bmatrix}
\mathcal{O} & \cdots & \mathcal{O} \\
\cdots & \mathcal{O} & \cdots \\
\mathcal{O} & \cdots & \mathcal{O}
\end{bmatrix}
\begin{bmatrix}
\mathcal{O} & \cdots & \mathcal{O} \\
\cdots & \mathcal{O} & \cdots \\
\mathcal{O} & \cdots & \mathcal{O}
\end{bmatrix}$$

These matrices are also the matrix representations with respect to the basis $B'$ of the stabilizer of $C'$.

Now it suffices to note that if $A$ is the transition matrix from the basis $B$ to $B'$, then multiplying $A$ on the left by matrices of the form (3.4) corresponds to acting on $B$ with an element that fixes $C$, and multiplying $A$ on the right with such a matrix corresponds to acting on $B'$ by an element that fixes $C'$. The matrices $(I-J_k)$ in the proof of Proposition A.9 are such matrices if we choose the element $a_{ij}$ of $A$ such that $i$ is maximal, and hence by using the same method as in that proof, we can reduce $A$ to a monomial matrix and in the process change $B$ to a basis $B''$ so that both $C$ and $C'$ are in the apartment $\Sigma_{B''}$.

(B2) Let the two apartments be $\Sigma_B$ and $\Sigma_{B'}$, where $B = \{b_1, \ldots, b_n\}$ and $B' = \{b'_1, \ldots, b'_n\}$ are two bases of $\mathbb{F}^n$, and let $C$ be a common chamber. Define two bijections $\phi_B : \Sigma_B \rightarrow \mathbb{Z}^{n-1}$ and $\phi_{B'} : \Sigma_{B'} \rightarrow \mathbb{Z}^{n-1}$ defining $\phi$ as in the proof of (B0). Let $w \in \mathbb{Z}^{n-1} \times \mathbb{Z}^n$ be the element mapping $\phi_B(C)$ to the fundamental chamber of $\mathbb{Z}^{n-1}$, i.e. $(0, \ldots, 0) \sim \cdots \sim (1, \ldots, 1)$, and let $v \in \mathbb{Z}^{n-1} \times \mathbb{Z}^n$ be the element that maps $\phi_{B'}(C)$ to $(0, \ldots, 0) \sim \cdots \sim (1, \ldots, 1)$. Then $\phi_{B'} \circ v^{-1} \circ w \circ \phi_B$ is a bijection from $\Sigma_B$ to $\Sigma_{B'}$ that fixes the intersection $\Sigma_B \cap \Sigma_{B'}$.

Since the geometric realization of any apartment in this building is homeomorphic to $\mathbb{R}^n$, this building is said to be affine or Euclidean.

If $n = 2$, then the building $\Delta$ is a regular infinite tree. It can be shown that each vertex of this tree has valency equal to $|\kappa| + 1$, where $\kappa = \mathcal{O}/\pi \mathcal{O}$. Figure 3.2 shows the building for the group $\text{SL}_2(\mathbb{Q})$, where $\mathbb{Q}$ is valued by the valuation $\omega_2$. 
3.2 Strongly transitive groups

Definition 3.5. A group action $G$ on a building $\Delta$ is strongly transitive if it acts transitively on the pair $(\Sigma, C)$, where $\Sigma$ is any apartment of $\Delta$ and $C$ is any chamber of $\Sigma$. That is, if $\Sigma$ and $\Sigma'$ are any two apartments, and $C \in \Sigma$ and $C' \in \Sigma'$ are two chambers, there is an element $g \in G$ with $\Sigma' = g\Sigma$ and $C' = gC$.

We will now look at what properties a group acting strongly transitively on a building has. As it turns out, such a group will actually contain all the information of the building, and it is possible to construct the building from the group, much in the same way as a Coxeter complex can be constructed from the Coxeter group.

So, for the remainder of this section, suppose we have a thick building $\Delta$ with a fundamental apartment $\Sigma$ and a fundamental chamber $C$, and suppose that the group $G$ acts strongly transitively on $\Delta$ and that this action is type-preserving. Three subgroups of $G$ are important, namely

$$B = \{g \in G \mid gC = C\},$$
$$N = \{g \in G \mid g\Sigma = \Sigma\}$$
and
$$T = \{g \in G \mid g \text{ fixes } \Sigma \text{ pointwise}\}.$$

As the apartments are all Coxeter complexes, we have a Coxeter group $W$ acting on the apartments, which is generated by a set $S$ of elements of order 2. There is a natural homomorphism $\alpha$ from $N$ to $W$ that is obtained by restricting the action of $N$ to $\Sigma$. The kernel of $\alpha$ is precisely $T$, and so $W \simeq N/T$. Note that this means that we can view an element $w \in W$ as the corresponding coset $wT$, for some representative $\tilde{w} \in N$. Furthermore, $T = B \cap N$. To see this, clearly any element in $T$ is in both $B$ and $N$. On the other hand, suppose we have an element $g$ that is both in $B$ and $N$. As $G$ is type-preserving, $g$ fixes all chambers of $\Sigma$ adjacent to $C$. Extending this argument shows that $g$ fixes all of $\Sigma$, and hence is in $T$.

Now, the canonical labeling of $\Sigma(W, S)$ can be transferred to $\Sigma$ by the above isomorphism, and this labeling can be extended to all of $\Delta$ by using the type preserving action of $G$.

Proposition 3.6. The function $\phi$ from the set of left cosets of $B$ to the chambers of $\Delta$ defined by $\phi(gB) = gC$ is a bijection.

Proof. First we need to show that $\phi$ is well defined, that is, independent on the choice of coset representative. Suppose there are two elements $g, g' \in G$ with $gB = g'B$. Then there are two elements $b, b' \in B$ with $gb = g'b'$, or $g = g'b'b^{-1}$. But then

$$gC = g'b'b^{-1}C = g'b'C = g'C$$

as both $b'$ and $b^{-1}$ are in $B$. Thus $\phi(gB) = \phi(g'B)$, and $\phi$ is well defined.

As $G$ is transitive on the set of chambers of $\Delta$, any chamber of $\Delta$ can be written as $gC$ for some $g \in G$. So $\phi(gB) = gC$, and $\phi$ is surjective.

Figure 3.2: The affine building for the group $\text{SL}_2(\mathbb{Q})$, where $\mathbb{Q}$ is valuated by $\omega_2$. 
3.2. STRONGLY TRANSITIVE GROUPS

Let $A$ be the face of $C$ of type $S \setminus S'$ for some $S' \subseteq S$, and let $P_{S'}$ be the stabilizer of $A$. As before, let $h \in P_{S'}$, and let $\Sigma'$ be an apartment containing both $C$ and $hC$. Let $b \in B$ be an element with $b\Sigma' = \Sigma$. Then, as both $B$ and $P_{S'}$ stabilize $A$, the face of $hC$ of type $S \setminus S'$ is $A$. The stabilizer of $A$ in $\Sigma$ is $\langle S' \rangle$, and hence $bhC = sC$ for some $s \in (S')$. That is, $bh \in \langle S' \rangle B$ or equivalently $h \in b^{-1} (S') B$. It follows that $P_{S'} \subseteq B \langle S' \rangle B$, and as the reverse inequality is obvious we have shown that

$$P_s = B \cup BsB = B \{1, s\} B = B \langle s \rangle B.$$

Lemma 3.7. $B \langle S' \rangle B$ is a subgroup of $G$ for all $S' \subseteq S$.

Proof. Clearly $B \langle S' \rangle B$ contains the identity, and if $g \in B \langle S' \rangle B$ then $g^{-1} \in B \langle S' \rangle B$. Thus we only have to show that $B \langle S' \rangle B$ is closed. But $B \langle S' \rangle B$ is precisely the stabilizer of the face of $C$ of type $S \setminus S'$. Hence, if $g, g' \in B \langle S' \rangle B$, then clearly $gg'$ stabilizes the face of $C$ of type $S \setminus S'$ as well, and so $gg' \in B \langle S' \rangle B$. □

Definition 3.8. The subgroups of $G$ of the form $B \langle S' \rangle B$ are called the special subgroups of $G$. The cosets of the special subgroups are called special cosets.

Proposition 3.9. Let $\Delta$ be a building, $\Sigma$ be the fundamental apartment of $\Delta$, $C \subseteq \Sigma$ be the fundamental chamber of $\Delta$, and $G$ be a group acting on $\Delta$ whose action is strongly transitive and type preserving. Then the set of special cosets of $G$ ordered by opposite inclusion, denoted by $\Delta(G, B)$, is isomorphic to the simplicial complex $\Delta$.

Proof. We define a map $\phi$ from $\Delta$ to $\Delta(G, B)$ by

$$\phi(g' \langle S \rangle B) = g' B \langle S' \rangle B,$$

where $g' \langle S \rangle B$ is the face of type $S \setminus S'$ of the chamber $g' C$. First, we show that this map is well-defined. Suppose $g' \langle S \rangle B = g'' \langle S \rangle B$. We must show that $g' B \langle S' \rangle B = g'' B \langle S' \rangle B$. But $g' \langle S \rangle B = g'' \langle S \rangle B$ implies $A_{S'} = g'^{-1} g'' A_{S'}$, and hence $g'^{-1} g'' \in B \langle S' \rangle B$. So, $B \langle S' \rangle B = g'^{-1} g'' B \langle S' \rangle B$, or $g' B \langle S' \rangle B = g'' B \langle S' \rangle B$ and $\phi$ is well-defined.
φ is clearly surjective. To show that φ is injective, suppose \( g'B \langle S' \rangle = g''B \langle S'' \rangle \). Then \( B \langle S' \rangle B = g'^{-1}g''B \langle S'' \rangle B \). But since 1 ∈ \( B \langle S' \rangle B \) it follows that \( (g'^{-1}g'')^{-1} \in B \langle S' \rangle B \), and so \( g'^{-1}g'' \in B \langle S'' \rangle B \) as \( B \langle S'' \rangle B \) is a group. But then it follows that

\[
B \langle S' \rangle B = g'^{-1}g''B \langle S'' \rangle B = B \langle S'' \rangle B,
\]

and so \( S' = S'' \). Now, \( g'B \langle S' \rangle B = g''B \langle S'' \rangle B \) implies \( g'^{-1}g'' \in B \langle S' \rangle B = P_{S'} \) and hence \( A_{S'} = g'^{-1}g''A_{S''} \), and so \( g'A_{S'} = g''A_{S'} \). Thus, \( g'B \langle S' \rangle B = g''B \langle S'' \rangle B \) implies \( g'A_{S'} = g''A_{S''} \), and so φ is injective.

To complete the proof we must show that φ preserves the poset structure. That is, if \( g'A_{S'} \leq g''A_{S''} \) then \( g'B \langle S' \rangle B \supseteq g''B \langle S'' \rangle B \). Suppose \( g'A_{S'} \leq g''A_{S''} \). Then \( g'^{-1}g''A_{S'} \leq A_{S'} \leq C \).

Since \( G \) is type-preserving, we have \( g'^{-1}g' A_{S'} = A_{S'} \). But then \( A_{S'} \leq A_{S''} \), and so \( B \langle S' \rangle B \supseteq B \langle S'' \rangle B \). Also, \( g'^{-1}g' \in B \langle S' \rangle B \), and so

\[
g'^{-1}g'B \langle S' \rangle B = B \langle S' \rangle B \supseteq B \langle S'' \rangle B,
\]

and hence \( g'B \langle S' \rangle B \supseteq g''B \langle S'' \rangle B \) as required. \( \square \)

### 3.3 The spherical building of \( \text{GL}_n(F) \)

Let \( F \) be a field and let \( V \) be the \( n \)-dimensional vector space \( F^n \). Let \( \Delta \) be the flag complex of subspaces in \( V \), as defined in Example 3.3. As we saw in Example 3.3, this is a building, where the apartments are the subcomplexes given by the frames.

The action of the group \( \text{GL}_n(F) \) on \( V \) can be extended to \( \Delta \) in a natural way, by defining

\[
g(V_1 \subset \cdots \subset V_m) = g(V_1) \subset \cdots \subset g(V_m)
\]

for \( g \in \text{GL}_n(F) \). Clearly, this action is type-preserving.

We will now show that the action of \( \text{GL}_n(F) \) on \( \Delta \) is strongly transitive. So, let \( \Sigma \) and \( \Sigma' \) be any two apartments, and let \( C \) be any chamber of \( \Sigma \) and \( C' \) any chamber of \( \Sigma' \). We must find an element \( g \in \text{GL}_n(F) \) with \( g\Sigma = \Sigma' \) and \( gC = C' \).

Let \( F = \{L_1, \ldots, L_n\} \) be a frame of \( \Sigma \) with \( C = L_1 \subset L_1 \oplus L_2 \subset \cdots \subset L_1 \oplus \cdots \oplus L_{n-1} \), and let \( F' = \{L_1', \ldots, L_n'\} \) be a frame of \( \Sigma' \) with \( C' = L_1' \subset L_1' \oplus L_2' \subset \cdots \subset L_1' \oplus \cdots \oplus L_{n-1}' \). Now it is easy to find the element \( g \). Let \( e_i \) be a non-zero vector in \( L_i \), let \( e_i' \) be a non-zero vector in \( L_i' \), and define \( g \) by \( g(e_i) = e_i' \). Then clearly \( g(\Sigma) = \Sigma' \) and \( g(C) = C' \), and hence \( \text{GL}_n(F) \) acts strongly transitively on \( \Delta \).

Now, let \( b_1, \ldots, b_n \) be the standard basis of \( V \), let the fundamental apartment \( \Sigma \) be given by the frame \( F = \{b_1, b_2, \ldots, b_n\} \) and let the fundamental chamber \( C \) be given by

\[
C = \langle b_1 \rangle \subset \langle b_1, b_2 \rangle \subset \cdots \subset \langle b_1, \ldots, b_{n-1} \rangle.
\]

We will now determine the subgroups \( B \), \( N \), and \( T \). The stabilizer of \( C \) (i.e. \( B \)) is simply the upper triangular subgroup of \( \text{GL}_n(F) \). The stabilizer of \( \Sigma \) is the monomial subgroup of \( \text{GL}_n(F) \), i.e. the set of invertible matrices with one non-zero entry in each row and column. Finally, the subgroup \( T = B \cap N \) is the set of invertible diagonal matrices.

So, what about the group \( W = N/T \)? The action of \( N \) permutes the coordinates, and multiplies each of them by a constant. Quotienting out by \( T \), all that is left is a permutation, so \( W \) is isomorphic to the permutation group on \( n \) letters, \( W \simeq S_n \). This was expected, as we saw in Example 3.3 that the apartments of \( \Delta \) are isomorphic to the Coxeter complex of \( S_n \).

Since this building is spherical, it is called the spherical building of the group \( \text{SL}_n(F) \).

### 3.4 The affine building of \( \text{SL}_n(F) \), where \( F \) is a field with a discrete valuation

In this section we follow the notation of Example 3.4. Let \( \Delta \) be the building defined in Example 3.4 and consider the action of the group \( \text{SL}_n(F) \) on \( \Delta \). This action is strongly transitive, since for any lattice \( L \) and \( g \in \text{SL}_n(F) \) we have

\[
\lambda(gL) = \omega \circ \det(g) + \lambda(L) \equiv 0 + \lambda(L) \equiv \lambda(L) \quad (\text{mod } n).
\]
For the rest of this section, any matrix representation of an element of \( \text{SL}_n(F) \) is with respect to the standard basis \( B' = \{b_1, \ldots, b_n\} \) (we need the prime to distinguish this basis from the subgroup \( B \) of \( F^n \)).

Let the fundamental apartment \( \Sigma \) be \( \Sigma' \), and let the fundamental chamber be

\[
C = [[b_1, \ldots, b_n]] \sim [[\pi b_1, b_2, \ldots, b_n]] \sim \ldots \sim [[\pi b_1, \ldots, \pi b_{n-1}, b_n]].
\]

We will show that \( \text{SL}_n(F) \) acts strongly transitively on \( \Delta \). The subgroup \( N < \text{SL}_n(F) \) that stabilizes \( \Sigma \) is the monomial subgroup of \( \text{SL}_n(F) \), with respect to the basis \( B' \). The subgroup \( B < \text{SL}_n(F) \) that stabilizes \( C \) is, as we saw in Example 3.4, precisely those elements of \( \text{SL}_n(F) \) with matrices of the form

\[
\begin{bmatrix}
O & \pi O \\
\vdots & \ddots \\
O & \cdots & O
\end{bmatrix}.
\]

As \( \text{GL}_n(F) \) is transitive on the set of bases of \( F^n \), it is transitive on the set of apartments of \( \Delta \) as well. The subgroup of \( \text{GL}_n(F) \) that stabilizes \( \Sigma \) is, as in the case of \( \text{SL}_n(F) \), the monomial subgroup. Let \( \Sigma' \) be any apartment, and let \( g \in \text{GL}_n(F) \) be an element with \( g\Sigma = \Sigma' \). Now, let \( h \) be an element of the monomial subgroup of \( \text{GL}_n(F) \) with \( \det(h) = (\det(g))^{-1} \). Then \( \det(gh) = 1 \), so \( gh \in \text{SL}_n(F) \), and

\[
gh\Sigma = g\Sigma = \Sigma'.
\]

Hence \( \text{SL}_n(F) \) is transitive on the set of apartments of \( \Delta \).

We now show that \( N \) is transitive on the set of chambers of \( \Sigma \). Note that for every element \( g \) of \( N \) there are unique integers \( k_1, \ldots, k_n \) with

\[
g = \begin{bmatrix}
\pi^{k_1} & 0 \\
& \ddots \\
& & \pi^{k_n}
\end{bmatrix} s,
\]

where \( s \) is a monomial matrix with entries in \( O^* \). Note also that, since \( g \in \text{SL}_n(F) \), \( k_1 + \cdots + k_n = 0 \). The action of such an element \( s \) on a lattice is hence just a permutation of the basis, and hence \( s \) can be identified with its corresponding element of \( S_n \). Also, the map \( \theta : N \rightarrow \mathbb{Z}^{n-1} \ltimes S_n \) defined by

\[
\theta : \begin{bmatrix}
\pi^{k_1} & 0 \\
& \ddots \\
& & \pi^{k_n}
\end{bmatrix} s \mapsto [x, s],
\]

where \( x = (2k_1 + k_2 + \cdots + k_{n-1}, \ldots, k_1 + \cdots + k_{n-2} + 2k_{n-1}) \) (compare with the basis \( \{a_1, \ldots, a_n\} \) of Example 2.17), is a homomorphism. Let \( \phi \) be the bijection from the vertices of \( \Sigma \) to \( \mathbb{Z}^{n-1} \) defined by \( \begin{bmatrix}
\pi^{k_1}b_1, \ldots, \pi^{k_{n-1}}b_{n-1}, b_n
\end{bmatrix} \mapsto (l_1, \ldots, l_{n-1}) \), and extend \( \phi \) to simplices. Now, if \( g \in N \) and \( A \in \Sigma \), then \( \phi \) and \( \theta \) satisfy

\[
\phi(gA) = \theta(g)\phi(A).
\]

Since \( \mathbb{Z}^{n-1} \ltimes S_n \) acts transitively on the set of chambers of the chamber complex of Example 1.19, it follows that \( N \) acts transitively on the set of chambers of \( \Sigma \).

Since \( \text{SL}_n(F) \) is transitive on the set of apartments of \( \Delta \), and \( N \) is transitive on the set of chambers of \( \Sigma \), it follows that \( \text{SL}_n(F) \) is strongly transitive on \( \Delta \). To see this, let \( \Sigma' \) and \( \Sigma'' \) be any two apartments of \( \Delta \), and let \( C' \in \Sigma' \) and \( C'' \in \Sigma'' \) be any two chambers in these apartments. Now, since \( \text{SL}_n(F) \) is transitive on the set of apartments of \( \Delta \) there are elements \( g' \), \( g'' \in \text{SL}_n(F) \) with \( g'\Sigma = \Sigma' \) and \( g''\Sigma = \Sigma'' \). Also, since \( N \) is transitive on the set of chambers of \( \Sigma \), there are elements \( h', h'' \in N \) with \( h'C = g'^{-1}C' \) and \( h''C = g''^{-1}C'' \). Hence, the element \( g''h'h^{-1}g'^{-1} \) maps \( \Sigma' \) to \( \Sigma'' \) and \( C' \) to \( C'' \), as required.

Finally, we note that the subgroup \( T < \text{SL}_n(F) \) that fixes \( \Sigma \) pointwise is the diagonal subgroup of \( \text{SL}_n(O) \). Noting that this is precisely the kernel of \( \theta \) defined above, we conclude that \( N/T \simeq \mathbb{Z}^{n-1} \ltimes S_n \), as expected.

Since this building is affine, it is called the affine building of \( \text{SL}_n(F) \).
Chapter 4

The tight span of an $\mathbb{R}$-tree

In this chapter we motivate the definition of the tight span of a tree, and explore some of its properties.

4.1 $\mathbb{R}$-trees

**Definition 4.1.** A metric space $(X, d)$ is called an $\mathbb{R}$-tree if it satisfies the following properties:

1. (RT1) For all $p, q \in X$ there is a unique isometry $\psi_{pq} : [0, d(p, q)] \hookrightarrow X$ such that $\psi_{pq}(0) = p$ and $\psi_{pq}(d(p, q)) = q$.
2. (RT2) All continuous injective maps $[0, 1] \hookrightarrow X, t \mapsto p_t$ satisfy $d(p_0, p_1) = d(p_0, p_t) + d(p_t, p_1)$ for all $t \in [0, 1]$.

For any two points $p, q \in X$, the set $\psi_{pq}([0, d(p, q)])$ is called the path from $p$ to $q$ (or equivalently, the path from $q$ to $p$), and is denoted by $\langle p, q \rangle$.

(RT1) and (RT2) together imply that for any two points in $X$, there is a unique path between them (recall the definition of a combinatorial tree).

**Example 4.2** Let $T = (V, E)$ be a weighted tree, i.e. a graph theoretical tree with weight function $w : E \to \mathbb{R}$. Then we can construct a natural $\mathbb{R}$-tree from $T$ by regarding each edge $e \in E$ as being a line isometric to the interval $[0, w(e)]$.

**Example 4.3** Endow $\mathbb{R}^n$ with the metric $d$ given by

$$d(x, y) = \begin{cases} \|x - y\| & \text{if there is a } t \in \mathbb{R}_+ \text{ with } x = ty, \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

That is, any continuous curve in $\mathbb{R}^n$ must move along a ray from the origin (see Figure 4.1). Then $(\mathbb{R}^n, d)$ is an $\mathbb{R}$-tree.

![Figure 4.1](image)

**Example 4.4** Let $X \subset \mathbb{R}^2$ be the set

$$X = \{(x, 0) \mid x \in \mathbb{R}_{\geq 0}\} \cup \{(n, y) \mid n \in \mathbb{Z}_{\geq 0}, y \in \mathbb{R}_{\geq 0}\}$$
(see Figure 4.2), with the shape of an ‘infinite comb’, and let
\[ d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + y_1 + y_2. \]
That is, the distance between two points in \( X \) is the length along the lines of \( X \). Then it is easy to see that \((X, d)\) is an \( \mathbb{R} \)-tree.

![Figure 4.2: The \( \mathbb{R} \)-tree defined in Example 4.4.](image)

Every \( \mathbb{R} \)-tree \((X, d)\) satisfies the **four-point condition**. That is, any four points \( p, q, r, s \in X \) satisfy the inequality
\[ d(p, q) + d(r, s) \leq \max \left\{ d(p, r) + d(q, s), d(p, s) + d(q, r) \right\}. \]
This follows from the fact that the two greater sums are equal (see Figure 4.3).

![Figure 4.3: The most general configuration of four points in an \( \mathbb{R} \)-tree. It is easy to see that the metric of an \( \mathbb{R} \)-tree must satisfy the four point condition.](image)

This motivates the following definition.

**Definition 4.5.** A metric space is said to be **treelike** if it satisfies the four-point condition.

For example, it is obvious that the metric on the vertices of a weighted tree induced by the weight function is treelike. However, this metric space is *not* an \( \mathbb{R} \)-tree, though it is easy to construct an \( \mathbb{R} \)-tree into which it is isometrically embeddable using the method described in Example 4.2. It can be shown that a metric space is embeddable into an \( \mathbb{R} \)-tree if and only if it is treelike. We will not show this, but a proof can be found in [5].

### 4.2 The tight span of a finite tree

Consider the \( \mathbb{R} \)-tree \((X, d)\) obtained from a weighted finite tree. Let \( E \) be the set of points in \( X \) corresponding to the leaves of the original tree. Now, note that each point \( p \in X \) is uniquely identified by its distance to the points of \( E \). Hence, each point \( p \in X \) corresponds to a function \( f_p : E \to \mathbb{R} \) given by
\[ f_p(e) = d(p, e). \]

Now, if \( e, e' \in E \) are such that \( p \in \langle e, e' \rangle \), then (see Figure 4.4a)
\[ f_p(e) + f_p(e') = d(e, e'). \]

On the other hand, if \( e, e' \in E \) are such that \( p \notin \langle e, e' \rangle \), then (see Figure 4.4b)
\[ f_p(e) + f_p(e') > d(e, e'). \]
In either case, suppose \( f_p \) satisfies

\[
 f_p(e) \geq d(e, e') - f_p(e').
\]

Since, given any \( e \in E \), there is at least one \( e' \in E \) with \( p \in \langle e, e' \rangle \), we conclude that

\[
 f_p(e) = \max_{e' \in E} \{d(e, e') - f_p(e')\}
\]

for all \( e \in E \).

This motivates the definition of the **tight span** of \((X, d)\) to be the set \( T_X \) defined by

\[
 T_X = \left\{ f : E \to \mathbb{R} \mid f(e) = \max_{e' \in E} \{d(e, e') - f(e')\} \text{ for all } e \in E \right\}.
\]

Note that all elements of \( T_X \) are positive functions on \( E \), since \( 2f(e) \geq d(e, e) = 0 \) for all \( e \in E \). Also, the elements of \( T_X \) are minimal, in the sense of the following lemma.

**Lemma 4.6.** Suppose \( f, g \in T_X \). If \( f(e) \leq g(e) \) for all \( e \in E \), then \( f = g \).

**Proof.** Suppose \( f(e) \leq g(e) \) for all \( e \in E \). Then, for all \( e \in E \),

\[
 f(e) \leq g(e) = \max_{e' \in E} \{d(e, e') - g(e')\} \leq \max_{e' \in E} \{d(e, e') - f(e')\} = f(e)
\]

and hence \( f = g \).  

As we have seen, the map \( p \mapsto f_p \) is an injection from \( X \) to \( T_X \). This map is actually a bijection from \( X \) to \( T_X \). To see this, let \( f \) be any element of \( T_X \). Let \( e_1, e_2 \in E \) be any elements with \( f(e_1) + f(e_2) = d(e_1, e_2) \). Now, let \( p \in \langle e_1, e_2 \rangle \) be the point with \( d(p, e_1) = f(e_1) \) (and hence \( d(p, e_2) = f(e_2) \)). It follows that \( f_p(e_1) = f(e_1) \) and \( f_p(e_2) = f(e_2) \).

We will show that \( f_p(e) = f(e) \) for all \( e \in E \). Let \( e \) be any element in \( E \). Then either \( p \in \langle e, e_1 \rangle \) or \( p \in \langle e, e_2 \rangle \) (see Figure 4.5a), so either \( f_p(e) = d(e, e_1) - f_p(e_1) \) or \( f_p(e) = d(e, e_2) - f_p(e_2) \). Also, \( f \) satisfies the inequalities

\[
 f(e) \geq d(e, e_1) - f(e_1)
\]

and

\[
 f(e) \geq d(e, e_2) - f(e_2).
\]

Thus either

\[
 f(e) \geq d(e, e_1) - f(e_1) = d(e, e_1) - f_p(e_1) = f_p(e)
\]

or

\[
 f(e) \geq d(e, e_2) - f(e_2) = d(e, e_2) - f_p(e_2) = f_p(e).
\]

In either case, \( f(e) \geq f_p(e) \), and so it follows by Lemma 4.6 that \( f = f_p \).

Define a metric on \( T_X \) by

\[
 d(f_p, f_q) = \max_{e \in E} \{|f_p(e) - f_q(e)|\}.
\]

We will now show that the map \( p \mapsto f_p \) is an isometry with respect to this metric on \( T_X \). Suppose \( p, q \in X, e \in E \). If there is a path in \( X \) containing \( p, q \) and \( e \), then (see Figure 4.5b)

\[
 |f_p(e) - f_q(e)| = |d(p, e) - d(q, e)| = |d(p, q)| = d(p, q).
\]
That is, two isometries are equal if they ’head off in the same direction’. An equivalence class is 
$$X$$
On the other hand, if there is no path in 
$$X$$
which is isometric to 
$$X$$
36

Figure 4.5: (a) Either
$$f_p(e) = d(e, e_1) - f_p(e_1)$$

If there is a path containing
$$p, q$$
and 
$$e$$
then
$$|f_p(e) - f_q(e)| = d(p, q).$$

(c) If there is no path containing
$$p, q$$
and 
$$e$$
then
$$|f_p(e) - f_q(e)| < d(p, q).$$

On the other hand, if there is no path in
$$X$$
containing
$$p, q$$
and 
$$e$$,
then (see Figure 4.5c)

$$|f_p(e) - f_q(e)| = |d(p, e) - d(q, e)| < d(p, q).$$

Hence,

$$d(f_p, f_q) = \max_{e \in E} \{ f_p(e) - f_q(e) \} = d(p, q).$$

So, using only the distances on the leaves of the tree
$$X$$
we have constructed a metric space
$$T_X$$
which is isometric to
$$X$$. In essence, we have reconstructed the tree
$$X$$.

### 4.3 The tight span of an infinite tree

**Definition 4.7.** An
$$\mathbb{R}$$
tree
$$X$$
is said to be
**infinite** if, for any
$$p \in X$$,
there is an isometric embedding
$$\psi : \mathbb{R} \to X$$
such that
$$\psi(0) = p.$$  

An infinite
$$\mathbb{R}$$
tree is thus an
$$\mathbb{R}$$
tree without leaves. Two examples of infinite
$$\mathbb{R}$$
trees are the ones defined in Example 4.3 and Example 4.4.

Suppose that we have an infinite
$$\mathbb{R}$$
tree
$$X$$,
and consider the set of isometries of
$$\mathbb{R}_{\geq 0}$$
into
$$X$$. Define an equivalence relation on the set of such isometries by saying that two isometries
$$\psi$$
and
$$\phi$$
are equivalent if there is an
$$\alpha \in \mathbb{R}_{\geq 0}$$
and
$$\beta \in \mathbb{R}, \alpha + \beta \geq 0$$,
such that
$$\psi(t) = \phi(t + \alpha)$$
for all
$$t \geq \beta$$. That is, two isometries are equal if they ’head off in the same direction’.
An equivalence class is called an **end** of
$$X$$. Let
$$E$$
be the set of ends of
$$X$$.

Note that given any point
$$p \in X$$
and end
$$e \in E$$
there is a representative
$$\psi_p \in e$$
such that
$$\psi_p(0) = p.$$ The **path** from
$$p$$
to
$$e$$
(written
$$\langle p, e \rangle$$
) is the path
$$\psi_p(\mathbb{R}_{\geq 0})$$. Also, it is easy to see that for any two distinct ends
$$e, e' \in E$$,
we can find a point
$$p \in X$$
such that the representatives
$$\psi_p \in e$$
and
$$\psi_p \in e'$$
satisfy
$$\langle p, e \rangle \cap \langle p, e' \rangle = \{ p \}.$$  
The **path** from
$$e$$
to
$$e'$$
is the set
$$\langle e, e' \rangle = \langle p, e \rangle \cup \langle p, e' \rangle.$$  

If we fix
$$x \in X$$,
we can define a map
$$v : E \times E \to \mathbb{R} \cup \{-\infty\}$$
by
$$v(e, e') = -2 \sup \{ t \in \mathbb{R}_{\geq 0} \mid \psi_x(t) = \phi_x(t) \},$$
where
$$\psi_x \in e$$
is such that
$$\psi_x(0) = x$$
and
$$\phi_x \in e'$$
is such that
$$\phi_x(0) = x.$$ Such a map is called a **valuation** on the ends, and has the following properties:

(i) \( v(e, e') = v(e', e) \),

(ii) \( v(e, e') = -\infty \) if and only if \( e = e' \), and

(iii) \( v(e, f) + v(e', f') \leq \max \{ v(e, f') + v(e', f), v(e, e') + v(f, f') \} \) (the four-point condition).

Properties (i) and (ii) are obvious, and property (iii) follows by the same reasoning for the four-point condition holding for a tree metric; the two greater sums are always equal (see Figure 4.6).

The ends of Example 4.3 are very simple, each end can be identified with the corresponding ray from the origin. If we let
$$v$$
be defined relative to the origin, then it is given by

$$v(e, e') = \begin{cases} 
-\infty & \text{if } e = e', \\
0 & \text{if } e \neq e'.
\end{cases}$$


The ends of Example 4.4 are also easy to identify. There is one end \( e_\infty \) corresponding to the positive \( x \)-axis and one end \( e_n \) for each non-negative integer. If \( v \) is defined relative to the origin, then

\[
v(e_i, e_j) = \begin{cases} 
-\infty & \text{if } i = j, \\
-2\min\{i, j\} & \text{if } i \neq j,
\end{cases}
\]

where \( i, j \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \).

What is really interesting is that using only the ends \( E \) and the map \( v \) we can reconstruct the infinite \( \mathbb{R} \)-tree. The construction is almost identical to the one for finite \( \mathbb{R} \)-trees, but we use the ends instead of the leaves, and \( v \) instead of \( d \). Hence, the tight span of an infinite \( \mathbb{R} \) tree \( T \) with ends \( E \) is the set

\[
T_X = \left\{ f : E \to \mathbb{R} \mid f(e) = \max_{e' \in E} \{v(e, e') - f(e')\} \right\}
\]

with a metric \( d \) given by

\[
d(f, g) = \max_{e \in E} |f(e) - g(e)|.
\]

The identification of \( T_X \) with \( X \) is not as obvious as in the case of a finite \( \mathbb{R} \)-tree, so it will require a little more work.

Let \( p \) be any point in \( X \). We want to define a function \( f_p : E \to \mathbb{R} \) that corresponds to \( p \) in a natural way. As we have seen, for any end \( e \in E \) there is one representative \( \psi_p \in e \) such that \( \psi_p(0) = x \) and one representative \( \psi_p \in e \) such that \( \psi_p(0) = p \). Now, define \( f_p(e) \) to be the real number \( \alpha \) such that \( \psi_p(t) = \psi_x(t + \alpha) \) for all \( t > c \) for some \( c \in \mathbb{R} \) (see Figure 4.7).

We claim that this function is in \( T_X \). To see this, let \( e, e' \in E \) be any two ends, and let \( l \in \mathbb{R}_{\geq 0} \) be such that \( v(e, e') = -2l \). If \( p \in \langle e, e' \rangle \), then \( f_p(e) = -l + l' \) and \( f_p(e') = -l - l' \) for some \( l' \in \mathbb{R} \) (see Figure 4.8). Hence \( f_p(e) + f_p(e') = v(e, e') \). On the other hand, if \( e' \in E \) is an end such that \( p \not\in \langle e, e' \rangle \), then we get three cases (see Figure 4.9), but in all three cases we get \( f_p(e) + f_p(e') > v(e, e') \). We conclude that \( f_p \) satisfies

\[
f_p(e) = \max_{e' \in E} \{v(e, e') - f_p(e')\}
\]

for all \( e \in E \), and hence \( f_p \in T_X \). Using the same argument as in the finite case, each element of \( T_X \) can be uniquely identified with the function \( f_p \) for some \( p \in X \). Hence we have a bijection
CHAPTER 4. THE TIGHT SPAN OF AN ℝ-TREE

Figure 4.8: If $p \in \langle e, e' \rangle$, then $f_p(e) + f_p(e') = v(e, e')$.

Figure 4.9: If $p \notin \langle e, e' \rangle$, then $p$ can be positioned as either point $p, p'$ or $p''$ in the figure. In either case, $f_p(e) + f_p(e') > v(e, e')$.

$X \to T_X : p \mapsto f_p$. Also, again using the same argument as in the finite case, this bijection is easily seen to be an isometry.

Note that, given two distinct ends $e, e' \in E$, a point $p \in X$ is in $\langle e, e' \rangle$ if and only if $f_p(e) + f_p(e') = v(e, e')$.

**Example 4.8** Recall the infinite ℝ-tree of Example 4.3. We have seen that its ends are simply the rays from the origin. The elements of its tight span is also very easy to describe. For any point $p \in X$, the corresponding function $f_p \in T_X$ is given by

$$f_p(e) = \begin{cases} -\|p\| & \text{if } p \text{ is in the ray } e, \\ \|p\| & \text{otherwise.} \end{cases}$$
Chapter 5

Valuated matroids

In this chapter we will generalize the idea of a valuation on ends of infinite $\mathbb{R}$-trees, which will give us an object called a valuated matroid. It turns out that the tight span construction can be generalized to apply to valuated matroids as well. Also, the tight span of a particular valuated matroid is shown to be a geometric realization of the building for $\text{SL}_n(F)$, where $F$ is a discretely valuated field. Most of the definitions and results of this chapter are taken from [11].

5.1 Valuated matroids

Definition 5.1. A valuated matroid $M = (E, v)_n$ of rank $n$ consists of a non-empty set $E$ together with a map $v : E^n \to \{-\infty\} \cup \mathbb{R}$ satisfying

(VM0) For all $e_1 \in E$ there is a set of elements $e_2, \ldots, e_n \in E$ such that $v(e_1, \ldots, e_n) \neq -\infty$.

(VM1) For all $e_1, \ldots, e_n \in E$ and permutations $\sigma \in S_n$, $v$ satisfies

$$v(e_1, \ldots, e_n) = v(e_{\sigma(1)}, \ldots, e_{\sigma(n)}).$$

That is, $v$ is totally symmetric.

(VM2) $v(e_1, \ldots, e_n) = -\infty$ if $e_i = e_j$ for some $1 \leq i, j \leq n, i \neq j$.

(VM3) For all $e_1, \ldots, e_n, f_1, \ldots, f_n \in E$,

$$v(e_1, \ldots, e_n) + v(f_1, \ldots, f_n) \leq \max_{1 \leq i \leq n} \{v(e_1, \ldots, e_{i-1}, f_1, e_{i+1}, \ldots, e_n) + v(e_i, f_2, \ldots, f_n)\}.$$

The map $v$ is called a valuation on $E$. Every set $B = \{b_1, \ldots, b_n\}, b_i \in E$ satisfying $v(B) = v(b_1, \ldots, b_n) \neq -\infty$ is called a basis for $M$.

Property (VM0) is needed to prevent degenerate cases and properties (VM1), (VM2) and (VM3) are generalizations of the properties (i), (ii) and (iii) respectively, that a valuation on the ends of an infinite tree satisfy (see page 36). Hence the ends of an $\mathbb{R}$-tree together with the valuation on the ends is an example of a rank 2 valuated matroid.

Two elements $e, e' \in E$ are said to be parallel (written $e \sim e'$) if $v(e, e', e_3, \ldots, e_n) = -\infty$ for all $e_3, \ldots, e_n \in E$. Obviously, for a valuated matroid obtained from the ends of an infinite $\mathbb{R}$-tree no two distinct ends are parallel.

Example 5.2 Let $E$ be any non-empty set with at least $n$ elements, and define $v : E^n \to \{0\} \cup \{-\infty\}$ by

$$v(e_1, \ldots, e_n) = \begin{cases} -\infty & \text{if there are indices } i \text{ and } j \text{ with } e_i = e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $M = (E, v)_n$ is a valuated matroid. Such valuated matroids are called trivial. The ends of the tree defined in Example 4.3 with the valuation defined with respect to the origin is an example of a trivial valuated matroid.
We now want to define the tight span of a valuated matroid in a way similar to the way in which we defined the tight span of the ends of an infinite $\mathbb{R}$-tree. The tight span of the ends of an infinite $\mathbb{R}$-tree with valuation $v$ can be described as being the set of maps $f : E \to \mathbb{R}$ with

(i) $f(e) + f(e') \geq v(e, e')$ for all $e, e' \in E$,
(ii) for all $e \in E$ there is an element $e' \in E$ such that $f(e) + f(e') = v(e, e')$.

In analogy to this, we define the tight span of a valuated matroid $M = (E, v)_n$ (written $T_M$) to be the set of maps $f : E \to \mathbb{R}$ with

(i) $f(e_1) + \cdots + f(e_n) \geq v(e_1, \ldots, e_n)$ for all $e_1, \ldots, e_n \in E$,
(ii) For all $e_1 \in E$ there are elements $e_2, \ldots, e_n$ with $f(e_1) + \cdots + f(e_n) = v(e_1, \ldots, e_n)$.

In short,

$$T_M = \left\{ f : E \to \mathbb{R} \mid f(e) = \max_{e_2, \ldots, e_n \in E} \left\{ v(e, e_2, \ldots, e_n) - \sum_{i=2}^n f(e_i) \right\} \right\}.$$

As in the case of the tight span of a finite $\mathbb{R}$-tree, the elements of $T_M$ are minimal in the following sense.

**Lemma 5.3.** Suppose $f, g \in T_M$. If $f(e) \leq g(e)$ for all $e \in E$, then $f = g$.

**Proof.** Suppose that $f(e) \leq g(e)$ for all $e \in E$. Then

$$f(e) \leq g(e) = \max_{e_2, \ldots, e_n} \left\{ v(e, e_2, \ldots, e_n) - \sum_{i=2}^n g(e_i) \right\} \leq \left\{ v(e, e_2, \ldots, e_n) - \sum_{i=2}^n f(e_i) \right\} = f(e),$$

and hence $f(e) = g(e)$. □

We will soon prove a proposition that characterizes the elements of $T_M$ in a very nice way, but to prove it we need the following rather technical lemma.

**Lemma 5.4.** Suppose $e_1, \ldots, e_n, e'_1, \ldots, e'_r \in E$ for some $r, 1 \leq r \leq n$. Then

$$v(e'_1, \ldots, e'_r, e_{r+1}, \ldots, e_n) + (r-1)v(e_1, \ldots, e_n) \leq \max_{\sigma \in S_r} \sum_{i=1}^r v(e'_i, \hat{e}_{\sigma(i)}),$$

where we use the notation $\hat{e}_i = e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n$.

**Proof.** The inequality is obvious in case $r = 1$. If $r > 1$, by (VM3) there is an index $i_1 \leq r$ with

$$v(e'_1, \ldots, e'_{i_1-1}, e_{i_1}, e'_{i_1+1}, \ldots, e'_r, e_{r+1}, \ldots, e_n) + v(e'_1, e_1, \ldots, e_{i_1}, e_{i_1+1}, \ldots, e_n).$$

Now, applying (VM3) again, there is an index $i_2 \leq r, i_2 \neq i_1$ with

$$v(e'_1, e'_{i_1-1}, e_{i_1}, e'_{i_1+1}, \ldots, e'_r, e_{r+1}, \ldots, e_n) + v(e_1, \ldots, e_n) \leq v(e'_1, \ldots, e'_{i_1-1}, e_{i_1}, e'_{i_1+1}, \ldots, e'_{i_2-1}, e_{i_2}, e'_{i_2+1}, \ldots, e'_r, e_{r+1}, \ldots, e_n) + v(e'_1, e_1, \ldots, e_{i_1}, e_{i_1+1}, \ldots, e_n).$$

By continuing in this manner, we get a list of distinct indices $i_1, \ldots, i_s$ with $i_j \leq r$ satisfying

$$v(e'_1, \ldots, e'_r, e_{r+1}, \ldots, e_n) + (r-1)v(e_1, \ldots, e_n) \leq \sum_{j=1}^r v(e'_j, \hat{e}_{i_j}).$$

Moreover, the function $\sigma$ defined by $\sigma(j) = i_j$ is clearly in $S_r$. □

**Proposition 5.5.** Suppose that $M = (E, v)_n$ is a valuated matroid. Let $B = \{b_1, \ldots, b_n\}$ be a basis for $M$ and let $H = \{ (t_1, \ldots, t_n) \mid t_i \in \mathbb{R}, \sum_{i=1}^n t_i = 0 \}$. Then $\Psi_B : H \to T_M; t \mapsto f_t$ is an injective map, where for $t = (t_1, \ldots, t_n)$, $f_t$ is given by

$$f_t(e) = \max_{1 \leq i \leq n} \left\{ v(e, \hat{b}_i) + t_i \right\} - \left( \frac{n-1}{n} \right) v(b_1, \ldots, b_n).$$
Hence 5.1. VALUATED MATROIDS

First, note that

Proof. Let 

In other words, for any 

Proof. For any basis 

Lemma 5.7. Let 

Hence 

For any basis 

Example 5.6 Let 

is defined

Note that in the case of infinite 

Lemma 5.7. Let 

Then there is some 

Now, since
it follows immediately that \( \sum_{i=1}^{n} f(b_i) = v(b_1, \ldots, b_n) \).

On the other hand, suppose \( f \in T_M \) with \( \sum_{i=1}^{n} f(b_i) = v(b_1, \ldots, b_n) \) for a base \( B = \{b_1, \ldots, b_n\} \). Let \( t_i = f(b_i) - \frac{1}{n} v(b_1, \ldots, b_n) \). Then \( \sum_{i=1}^{n} t_i = 0 \), so \( t = (t_1, \ldots, t_n) \in H \). Now, let \( f_t = \Psi_B(t) \in A_B \). Clearly, \( f_t(b_i) = f(b_i) \) for all \( 1 \leq i \leq n \), so

\[
    f_t(e) = \max_{1 \leq i \leq n} \left\{ v(e, b_i) - \sum_{j=1}^{n} f(b_j) \right\} 
    \leq \max_{e_2, \ldots, e_n \in E} \left\{ v(e, e_2, \ldots, e_n) - \sum_{i=2}^{n} f(e_i) \right\} = f(e)
\]

for all \( e \in E \). Hence by Lemma 5.3 \( f = f_t \in A_B \). \( \square \)

**Corollary 5.8.** Let \( M = (E, v)_n \) be a valuated matroid. Given any \( f \in T_M \) and \( b_1 \in E \), there are elements \( b_2, \ldots, b_n \) with \( B = \{b_1, \ldots, b_n\} \) is a basis for \( M \) and \( f \in A_B \).

In particular, if \( B \) is the set of bases for \( M \), then

\[
    T_X = \bigcup_{B \in B} A_B.
\]

**Proof.** It follows from the definition of the tight span that for any \( f \in T_X \) and \( b_1 \in E \) there are elements \( b_2, \ldots, b_n \in E \) with \( f(b_1) + \cdots + f(b_n) = v(b_1, \ldots, b_n) \). Since \( f(b_i) \in \mathbb{R} \) for all \( 1 \leq i \leq n \), we have \( v(b_1, \ldots, b_n) \neq -\infty \) and hence \( B = \{b_1, \ldots, b_n\} \) is a basis for \( M \). By Lemma 5.7 \( f \in A_B \). \( \square \)

In view of this corollary, it follows that \( T_X \) is the union of images of real vector spaces of dimension \( n - 1 \). It is of great interest to understand how the basis apartments for different bases intersect.

**Theorem 5.9.** Suppose \( M = (E, v)_n \) is a valuated matroid, \( B_0 = \{b_1, \ldots, b_n\} \) a basis for \( M \), \( b_0 \in E \setminus B_0 \), and \( I \subseteq \{1, \ldots, n\} \) with \( B_i = (B_0 \setminus \{b_i\}) \cup \{b_0\} \) a basis for \( M \) if and only if \( i \in I \).

Then \( A_{B_0} = \cup_{i \in I} (A_{B_0} \cap A_{B_i}) \), and for all \( i \in I \) we have

\[
    \Psi_{B_0}^{-1}(A_{B_0} \cap A_{B_i}) = \left\{ (t_1, \ldots, t_n) \in H \mid t_i + v(B_i) = \max_{j \in I} \{t_j + v(B_j)\} \right\}.
\]

**Proof.** For any \( f \in A_{B_0} \) we have \( f(b_i) = v(B_0) + t_i - \left( \frac{n-1}{n} \right) v(B_0) = t_i + \frac{1}{n} v(B_0) \) for all \( 1 \leq i \leq n \). Hence

\[
    f(b_0) = \max_{1 \leq i \leq n} \left\{ v(b_0, b_i) + t_i \right\} - \left( \frac{n-1}{n} \right) v(B_0)
    = \max_{1 \leq i \leq n} \left\{ v(B_i) + f(b_i) - \frac{1}{n} v(B_0) \right\} - \left( \frac{n-1}{n} \right) v(B_0)
    = \max_{i \in I} \left\{ v(B_i) + f(b_i) \right\} - v(B_0),
\]

and hence \( f(b_0) + v(B_0) = \max_{i \in I} \{v(B_i) + f(b_i)\} \). Now, suppose \( i \in I \) with \( f(b_0) + v(B_0) = f(b_i) + v(B_i) \). Since \( f \in A_{B_0} \) we have

\[
    v(B_i) = f(b_0) + v(B_0) - f(b_i) = f(b_0) + \sum_{j=1}^{n} f(b_j) - f(b_i) = \sum_{j=0}^{n} f(b_j),
\]

and so \( f \in A_{B_i} \). Hence \( A_{B_0} = \cup_{i \in I} (A_{B_0} \cap A_{B_i}) \). Also,

\[
    f(b_0) + v(B_0) = f(b_i) + v(B_i) = \max_{j \in I} \{f(b_j) + v(B_j)\}.
\]

Using the identity \( f(b_i) = t_i + \frac{1}{n} v(B_0) \) we get

\[
    t_i + v(B_i) = \max_{j \in I} \{t_j + v(B_j)\}.
\]

\( \square \)
5.1. VALUATED MATROIDS

Corollary 5.10. Under the assumptions of Theorem 5.9, $\Psi_{B_0}^{-1}(A_{B_0} \cap A_{B_1})$ is the union of at most $(n-1)$ closed half spaces on the form

$$H(i, j) = \{(t_1, \ldots, t_n) \in H \mid t_i - t_j \geq v(B_i) - v(B_j)\},$$

where $j \in I$.

Lemma 5.11. Let $M = (E, v)_n$ be a valuated matroid, an let $B, B'$ be two bases such that $|B' \setminus B| = d$ for some $d, 1 \leq d \leq n$. Then there are bases $B_0 = B, B_1, \ldots, B_d = B'$ with $|B_i \setminus B_{i-1}| = 1$ and $A_B \cap A_{B'} = \cap_{i=0}^d A_{B_i}$.

Proof. We will use induction on $d$. If $d = 1$, the lemma clearly holds. So, let $1 < d \leq n$ and assume the lemma is true for all $d' < d$. Suppose $b_1 \in B \setminus B'$ and $b'_1 \in B' \setminus B$ with

$$v(B) + v(B') \leq v((B \setminus \{b_1\}) \cup \{b'_1\}) + v((B' \setminus \{b'_1\}) \cup \{b_1\})$$

(b and $b'$ exist by (VM3)). Let $B_1 = (B \setminus \{b_1\}) \cup \{b'_1\}$ and $B'_1 = (B' \setminus \{b'_1\}) \cup \{b_1\}$. Now, let $f \in A_B \cap A_{B'}$. Then

$$\sum_{b \in B} f(b) + \sum_{b' \in B'} f(b') = v(B) + v(B') \leq v(B_1) + v(B'_1)$$

$$\leq \sum_{b \in B_1} f(b) + \sum_{b' \in B'_1} f(b') = \sum_{b \in B} f(b) + \sum_{b' \in B'} f(b').$$

It follows that $\sum_{b' \in B'_1} f(b') = v(B'_1)$. Thus $f \in A_{B'_1}$ and hence

$$A_B \cap A_{B'} \subseteq A_{B'_1}.$$

Since $|B \setminus B'_1| = d - 1 < d$, by the induction hypothesis there are bases $B_0 = B, B_1, \ldots, B_{d-1} = B'_1$ with $|B_i \setminus B_{i-1}| = 1$ for all $i, 1 \leq i \leq d - 1$, and $A_B \cap A_{B'_1} = \cap_{i=0}^{d-1} A_{B_i}$. Setting $B_d = B'$ we get

$$A_B \cap A_{B'} \subseteq A_B \cap A_{B'_1} \cap A_{B'} = \cap_{i=0}^{d-1} A_{B_i} \cap A_{B'} = \cap_{i=0}^d A_{B_i} \subseteq A_B \cap A_{B'_1} \cap A_{B_d} = A_B \cap A_{B'},$$

and hence $\cap_{i=0}^d A_{B_i} = A_B \cap A_{B'}$ as required.

Corollary 5.12. Let $B$ and $B'$ be two bases of a valuated matroid $M = (E, v)_n$. Then $A_B = A_{B'}$ if and only if for each $b \in B$ there is one element $b' \in B'$ that is parallel to $b$.

Proof. Suppose that $B' \setminus B = \{b'\}$ and $B \setminus B' = \{b\}$. Now, $A_B = A_{B'}$ if and only if $\Psi_{B_0}^{-1}(A_B \cap A_{B'}) = H$, so by Theorem 5.9, $A_B = A_{B'}$ if and only if $b$ is the only element of $B$ such that both $(B \setminus \{b\}) \cup \{b'\}$ and $(B' \setminus \{b'\}) \cup \{b\}$ are bases of $M$. That is,

$$v(b, b', b_3, \ldots, b_n) = -\infty$$

for all $b_3, \ldots, b_n \in B \cap B'$, $v(b, B \cap B') \neq -\infty$, and $v(B' \cap B') \neq -\infty$. Now, let $e_3, \ldots, e_n \in E$. Then, if we write $B \cap B' = \{b_2, \ldots, b_n\}$, it follows by (VM3) that either

$$v(b, b_2, \ldots, b_n) + v(b, b', e_3, \ldots, e_n) \leq v(b', b_2, \ldots, b_n) + v(b, b, e_3, \ldots, e_n) = -\infty$$

or

$$v(b, b_2, \ldots, b_n) + v(b, b', e_3, \ldots, e_n) \leq v(b, b_1, \ldots, b_{i-1}, b', b_{i+1}, \ldots, b_n) + v(b, b_1, e_3, \ldots, e_n) = -\infty.$$

In either case, we must have $v(b, b', e_3, \ldots, e_n) = -\infty$ and hence $b \sim b'$.

Using Lemma 5.11 and induction on $|B \setminus B'|$ the result now follows.

Corollary 5.13. If $B, B'$ are two bases of a valuated matroid of rank $n$ with $|B \setminus B'| = d$, then $\Psi_B^{-1}(A_B \cap A_{B'})$ is a closed convex set, bounded by at most $d(n-1)$ affine $(n-2)$-dimensional hyperplanes of $H$ on the form $t_i - t_j = c$, where $1 \leq i, j \leq n$ and $c \in \mathbb{R}$. 

5.2 The valued matroid \( M_n(F) \)

Let \( F \) be a field with a discrete valuation \( \omega \). Let \( E = F^n \setminus \{0\} \), and define a map \( v : E^n \to \mathbb{Z} \cup \{-\infty\} \) by

\[
v(e_1, \ldots, e_n) = \omega \circ \det(e_1, \ldots, e_n) = \omega \begin{vmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & \ddots & \vdots \\ e_{1n} & \cdots & e_{nn} \end{vmatrix},
\]

where \( e_i = (e_{i1}, \ldots, e_{in}) \). Then \( M_n(F) = (E, v)_n \) is a valued matroid of rank \( n \). \( M_n(F) \) clearly satisfies (VM0), (VM1) and (VM2), and it satisfies (VM3) in view of the following lemma.

**Lemma 5.14 (Grassmann-Plücker).** For all vectors \( e_1, \ldots, e_n, e'_1, \ldots, e'_n \in F^n \) the determinant satisfies

\[
det(e_1, \ldots, e_n) \cdot det(e'_1, \ldots, e'_n) = \sum_{i=1}^n det(e_1, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_n) \cdot det(e_i, e'_2, \ldots, e'_n).
\]

**Proof.** We consider the cases \( det(e_1, \ldots, e_n) = 0 \) and \( det(e_1, \ldots, e_n) \neq 0 \). If \( det(e_1, \ldots, e_n) = 0 \) then the left side of the equation is zero. Since we can write \( e_1 = \sum_{i=2}^n k_i e_i \) for some \( k_i \in F \), we have

\[
det(e'_1, e_2, \ldots, e_n) \cdot det(e_1, e'_2, \ldots, e'_n) = \sum_{i=2}^n k_i det(e'_1, e_2, \ldots, e_n) \cdot det(e_i, e'_2, \ldots, e'_n)
\]

and

\[
det(e_1, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_n) \cdot det(e_i, e'_2, \ldots, e'_n)
\]

\[
= \det \left( \sum_{j=2}^n k_j e_j \right) \cdot det(e_1, e'_2, \ldots, e'_n)
\]

\[
= \sum_{j=2}^n k_j det(e_j, e_2, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_n) \cdot det(e_i, e'_2, \ldots, e'_n)
\]

\[
= k_i det(e_1, e_2, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_n) \cdot det(e_i, e'_2, \ldots, e'_n)
\]

\[
= -k_i det(e'_1, e_2, \ldots, e_n) \cdot det(e_i, e'_2, \ldots, e'_n)
\]

for \( 2 \leq i \leq n \). Hence the sum on the right hand side of the equation is zero, and the identity holds.

On the other hand, if \( det(e_1, \ldots, e_n) \neq 0 \), then by the Cramer rule we can write \( \sum_{i=1}^n k_i e_i = e'_1 \), where

\[
k_i = \frac{\det(e_1, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_n)}{\det(e_1, \ldots, e_n)}
\]

Hence,

\[
\sum_{i=1}^n det(e_1, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_n) \cdot det(e_i, e'_2, \ldots, e'_n)
\]

\[
= \sum_{i=1}^n k_i det(e_1, \ldots, e_n) \cdot det(e_i, e'_2, \ldots, e'_n)
\]

\[
= det(e_1, \ldots, e_n) \cdot det(e'_1, \ldots, e'_n),
\]

which completes the proof. \( \square \)

Note that the bases of \( M_n(F) \) are precisely the bases of \( F^n \), and that two elements \( e, e' \) are parallel if and only if \( e = ke' \) for some \( k \in F^* \). Also, note that the image of \( F \) under \( v \) is \( \mathbb{Z} \cup \{-\infty\} \), hence \( M_n(F) \) is what is called a discrete valued matroid. For a general valued matroid, the set \( E \) need not have any structure at all. In this case however, it makes sense to ask whether the vector space structure of \( F^n \) gives \( M_n(F) \) any special structure. To make things easier, we extend the domain of \( v \) to \( (F^n)^n \) by setting \( v(0, e_2, \ldots, e_n) = -\infty \) for all \( e_i \in F^n \). We get the following three identities.
Lemma 5.15. Let \( e'_1, e_1, \ldots, e_n, \in E, k \in F \) and \( g \in \text{GL}_n(F) \). Then

(i) \( v(e_1 + e'_1, e_2, \ldots, e_n) \leq \max \{ v(e_1, \ldots, e_n), v(e'_1, e_2, \ldots, e_n) \} \),

(ii) \( v(k e_1, e_2, \ldots, e_n) = \omega(k) + v(e_1, \ldots, e_n) \), and

(iii) \( v(ge_1, \ldots, ge_n) = \omega \circ \det(g) + v(e_1, \ldots, e_n) \).

Proof. Using the multi-linearity and other properties of the determinant function, we have

\[
\begin{align*}
(i) & \quad v(e_1 + e'_1, e_2, \ldots, e_n) = \omega(\det(e_1 + e'_1, e_2, \ldots, e_n)) \\
& \quad = \omega(\det(e_1, \ldots, e_n) + \det(e'_1, e_2, \ldots, e_n)) \\
& \quad \leq \max \{ \omega(\det(e_1, \ldots, e_n), \omega(\det(e'_1, e_2, \ldots, e_n)) \} \\
& \quad = \max \{ v(e_1, \ldots, e_n), v(e'_1, e_2, \ldots, e_n) \},
\end{align*}
\]

(ii) \( v(k e_1, e_2, \ldots, e_n) = \omega(k) \det(e_1, e_2, \ldots, e_n) = \omega(k) + v(e_1, e_2, \ldots, e_n) \), and

(iii) \( v(ge_1, \ldots, ge_n) = \omega(\det(ge_1, \ldots, ge_n)) = \omega(\det(g) \det(e_1, \ldots, e_n)) \\
\quad = \omega \circ \det(g) + v(e_1, \ldots, e_n). \)

We will now describe the tight span \( T_M \) of \( M_n(F) \). We extend the domain of the elements of \( T_M \) to \( F^n \) by setting \( f(0) = -\infty \) for all \( f \in T_M \).

Lemma 5.16. For all \( f \in T_M, e \in E \) and \( k \in F^* \) we have

\[
f(ke) = \omega(k) + f(e).
\]

Proof.

\[
f(ke) = \max_{e_2, \ldots, e_n \in E} \left\{ v(ke, e_2, \ldots, e_n - \sum_{i=2}^{n} f(e_i)) \right\} \\
\quad = \max_{e_2, \ldots, e_n \in E} \left\{ v(ke, e_2, \ldots, e_n + \omega(k) - \sum_{i=2}^{n} f(e_i)) \right\} = \omega(k) + f(e).
\]

We know that every basis of \( F^n \) gives a basis apartment of \( T_M \). Also, we know by Corollary 5.13 that the intersection of two apartments is the image of a convex set in \( H \), bounded by hyperplanes on the form

\[
H(i, j; k) = \{(t_1, \ldots, t_n) \in H \mid t_i - t_j = k \},
\]

where \( 1 \leq i, j \leq n \). Since \( v(B) \in Z \) for all bases \( B \) of \( M_n(F) \), it follows that \( k \in Z \). We saw at the end of Chapter 2 that this is precisely the set of hyperplanes that generate the reflection group \( \mathbb{Z}^{n-1} \rtimes S_n \). Hence, \( T_M \) is the union of images of basis apartments, where the bases are the bases of \( F^n \) and each basis apartment in some sense has the structure of the Coxeter complex of the group \( \mathbb{Z}^{n-1} \rtimes S_n \). This strongly suggests that \( T_M \) is closely related to the building for \( \text{SL}_n(F) \). In fact, we will now show that \( T_M \) is a geometric realization of the building for \( \text{SL}_n(F) \).

Recall that the vertices of the Coxeter complex of \( \mathbb{Z}^{n-1} \rtimes S_n \) can be identified with the points \((t_1, \ldots, t_n) \in H \) with \( t_j \equiv t \pmod{1} \) for some \( 0 \leq t \leq (n-1) \) and all \( 1 \leq j \leq n \), and that these points are called the vertices of \( H \). The following lemma shows that the images of these points under \( \Psi_H \) have special properties.
Lemma 5.17. Let \( B \) be a basis for \( M_n(F) \). Let \( t \in H \) be a vertex of \( H \) and \( f = \Psi_B(t) \). Then there is some \( i \in \{0, \ldots, n-1\} \) with \( f(e) \equiv \frac{1}{n} \mod 1 \) for all \( e \in E \). Conversely, if \( f \in T_M \), there is some \( i \in \{0, \ldots, n-1\} \) with \( f(e) \equiv \frac{1}{n} \mod 1 \) for all \( e \in E \), then the pre-image \( \Psi_B^{-1}(f) \) is a vertex of \( H \) for all bases \( B \) with \( f \in A_B \).

Proof. Let \( f = \Psi_B(t) \) for some vertex \( t = (t_1, \ldots, t_n) \in H \) and basis \( B = \{b_1, \ldots, b_n\} \) of \( M_n(F) \). For all \( e \in E \) there is some \( 1 \leq j \leq n \) with \( f(e) = v(b_j, e) + f(b_j) - v(B) \). Hence

\[
n f(e) = nv(b_j, e) + nf(b_j) + nv(B) = nv(b_j, e) + nt_j + v(B) - nv(B),
\]

and so \( nf(e) \in \mathbb{Z} \). Also, \( v(B) \) does not depend on \( e \), \( nv(b_j, e) \in n\mathbb{Z} \) for all \( e \in E \) and \( nt_j = nt_k \) for all \( 1 \leq j, k \leq n \). Hence there is some \( i \in \{0, \ldots, n-1\} \) with \( f(e) \equiv \frac{i}{n} \mod 1 \) for all \( e \in E \).

Conversely, let \( f \in T_M \) be such that there is some \( i \in \{1, \ldots, n\} \) with \( f(e) \equiv \frac{i}{n} \mod 1 \) for all \( e \in E \), and let \( B = \{b_1, \ldots, b_n\} \) be a basis with \( f \in A_B \). Then \( t_j = f(b_j) - \frac{1}{n}v(B) \equiv \frac{i-v(B)}{n} \mod 1 \) for \( (t_1, \ldots, t_n) = \Psi_B^{-1}(f) \) and \( 1 \leq j \leq n \). Hence \( (t_1, \ldots, t_n) \in H \) is a vertex. \( \square \)

Motivated by the previous lemma, we say that an element \( f \in T_M \) is a vertex of type \( i \in \{0, \ldots, n-1\} \) if \( f(e) \equiv \frac{i}{n} \mod 1 \) for all \( e \in E \).

Now, let \( C_H = \{(t_1, \ldots, t_n) \in H | t_1 \leq \cdots \leq t_n \leq t_1 + 1\} \), the convex hull in \( H \) of the vertices of the fundamental chamber of the Coxeter complex of \( \mathbb{Z}^{n-1} \times S_n \). A set \( C \subseteq T_M \) is called a chamber if there is a basis \( B \) of \( M_n(F) \) and an element \( w \in \mathbb{Z}^{n-1} \times S_n \) with

\[
C = \Psi_B \circ w(C_H).
\]

If \( t_1, \ldots, t_r \in H \) are vertices of \( C_H \), then the image of the convex hull of \( t_1, \ldots, t_r \) under \( \Psi_B \circ w \), that is, the set

\[
\Psi_B \circ w \left( \left\{ \sum_{i=1}^{r} \alpha_i t_i \mid \alpha_i \geq 0, \sum_{i=1}^{r} \alpha_i = 1 \right\} \right),
\]

is called a simplex of rank \( r \) of \( T_M \). Two chambers of \( T_M \) are said to be adjacent if their intersection is a simplex of rank \( (n-1) \).

Define an incidence relation on the set of simplices of \( T_M \) by saying that two simplices are incident if one is a subset of the other. Given a basis \( B \), the map \( \Psi_B \), interpreted in the right way, is a simplicial isomorphism from the Coxeter complex of \( \mathbb{Z}^{n-1} \times S_n \) to the apartment \( A_B \).

Let \( B_0 \) be the standard basis of \( F^n \). Then \( A_0 = A_{B_0} \) is called the fundamental apartment of \( T_M \) and \( C_0 = \Psi_{B_0}(C_H) \) the fundamental chamber of \( T_M \).

We will now, through a series of lemmas, prove that \( T_M \) is a geometric realization of the building for the group \( SL_n(F) \).

Define an action of \( GL_n(F) \) on \( T_M \) by

\[
(gf)(e) = f(g^{-1}e) + \frac{1}{n} \omega \circ \det(g),
\]

for \( g \in GL_n(F), f \in T_M \) and \( e \in E \). Let \( f \in A_B, B = \{b_1, \ldots, b_n\} \). If we write \( gB = \{gb_1, \ldots, gb_n\} \), then by Lemma 5.15(iii) we have

\[
\sum_{i=1}^{n} (gf)(gb_i) = \omega \circ \det(g) + \sum_{i=1}^{n} f(b_i) = \omega \circ \det(g) + v(B) = v(gB),
\]

and, for \( e \in E \),

\[
(gf)(e) = f(g^{-1}e) + \frac{1}{n} \omega \circ \det(g)
= \max_{1 \leq i \leq n} \left\{ v(b_i, g^{-1}e) + f(b_i) \right\} - v(B) + \frac{1}{n} \omega \circ \det(g)
= \max_{1 \leq i \leq n} \left\{ v((g)b_i, e) - \omega \circ \det(g) + (gf)(gb_i) - \frac{1}{n} \omega \circ \det(g) \right\}
- v(gB) + \omega \circ \det(g) + \frac{1}{n} \omega \circ \det(g)
= \max_{1 \leq i \leq n} \left\{ v((g)b_i, e) + (gf)(gb_i) \right\} - v(gB).
\]
Hence $gf \in A_B$, so in particular $gf \in T_M$ for all $g \in \text{GL}_n(F)$ and $f \in T_M$. If $g$ is the identity, then $gf = f$ for all $f \in T_M$, so this is indeed an action on $T_M$. Note that this action induces an action on the set of apartments of $T_M$, where the action is given by $gA_B = A_{gf}$.

**Lemma 5.18.** The action of $\text{GL}_n(F)$ on $T_M$ is transitive on the set of vertices of $T_M$. The restriction of this action to $\text{SL}_n(F)$ preserves the type of the vertices.

*Proof.* Let $f, f' \in T_M$ be two vertices and let $B = \{b_1, \ldots, b_n\}$ and $B' = \{b'_1, \ldots, b'_n\}$ be two bases with $f \in A_B$ and $f' \in A_{B'}$. Since $f$ and $f'$ are vertices we have $f(b_i) - f(b_i') - f'(b_i') + f'(b_i') \in \mathbb{Z}$ for $1 \leq i \leq n$, and hence we can choose $x_i \in F^*$ such that $\omega(x_i) = f(b_i) - f(b_i) - f'(b_i') + f'(b_i')$ for all $1 \leq i \leq n$.

Let $g \in \text{GL}_n(F)$ be such that $gb_i = x_i b_i'$ for $1 \leq i \leq n$. By Lemma 5.15(iii) we have

$$\omega \circ \det(g) = v(gb_1, \ldots, gb_n) - v(b_1, \ldots, b_n)$$

$$= v(b'_1, \ldots, b'_n) + \sum_{i=1}^n \omega(x_i) - v(b_1, \ldots, b_n)$$

$$= v(b'_1, \ldots, b'_n) - v(b_1, \ldots, b_n) + \sum_{i=1}^n f(b_i) - \sum_{i=1}^n f'(b_i') = n (f(b_n) - f'(b'_n))$$

$$= -n((f(b_n) - f'(b'_n))$$

and hence for $1 \leq j \leq n$ we have

$$(gf)(b'_j) = f(g^{-1} b'_j) = \frac{1}{n} \omega \circ \det(g) = f(b_j) - \omega(x_j) + \frac{1}{n} \omega \circ \det(g)$$

$$= f(b_j) - (f(b_j) - f(b_n) - f'(b'_j) + f'(b'_n) - f(b_n) + f'(b'_n) = f'(b'_j).$$

Thus $gf = f'$, so the action of $\text{GL}_n(F)$ on $T_M$ is transitive on the set of vertices. That the action of $\text{SL}_n(F)$ preserves the vertex type is an immediate consequence of the fact that $\omega \circ \det(g) = 0$ for all $g \in \text{SL}_n(F)$. $\square$

**Lemma 5.19.** The action of $\text{SL}_n$ is transitive on the set of apartments of $T_M$. The stabilizer of the fundamental apartment is the subgroup $N$ of monomial matrices. The subgroup that fixes $A_0$ pointwise is the group $T$ of diagonal matrices, whose diagonal elements $g_i$ satisfy $\omega(g_i) = 0$.

*Proof.* By Corollary 5.12, $A_B = A_0$ if and only if the elements in $B$ are parallel to the elements of the standard basis. It follows immediately that the stabilizer of $A_0$ is $N$. Also, since $\text{GL}_n(F)$ is transitive on the set of bases and every monomial matrix fixes $A_0$, it follows that for every apartment $A_B$ there is an element $g \in \text{SL}_n(F)$ with $gA_0 = A_B$. Thus, $\text{SL}_n(F)$ acts transitively on the set of apartments of $T_M$.

To see that the subgroup that fixes $A_0$ pointwise is the group of diagonal matrices whose diagonal elements satisfy $\omega(g_i) = 0$, suppose that $g = (g_i) \in T$. Then for all $f \in A_0$, if $B_0 = \{b_1, \ldots, b_n\}$ is the standard basis of $F^n$, we have

$$(gf)(b_i) = f(g^{-1} b_i) = f(\frac{1}{g_i} b_i) = f(b_i) - \omega(g_i) = f(b_i)$$

for all $1 \leq i \leq n$ and hence $gf = f$ for all $f \in A_0$. On the other hand, suppose that $g \in N$ such that $gf = f$ for all $f \in A_0$. Write $g^{-1} = (g_i)$ with $g_\sigma(i) = a_i \in F^*$ and $g_\sigma(i,j) = 0$ for some $\sigma \in S_n$ and all $i \neq j$. Then

$$f(b_i) = (gf)(b_i) = f(g^{-1} b_i) = f(a_i b_\sigma(i)) = \omega(a_i) + f(b_\sigma(i))$$

for all $1 \leq i \leq n$ and all $f \in A_0$. Hence we must have $\omega(a_i) = 0$ for all $1 \leq i \leq n$ and $\sigma$ must be the identity. Hence $g \in T$. $\square$

**Lemma 5.20.** The action of $\text{SL}_n(F)$ induces an action on the set of simplices of $T_M$, by $gD = \{gf \mid f \in D\}$ for all simplices $D \subseteq T_M$. The action of $N$ is transitive on the set of chambers of the fundamental apartment. The stabilizer of the fundamental chamber is the subgroup $B$ of $\text{SL}_n(F)$ of matrices $g = (g_i)$ with $\omega(g_i) \leq 0$ if $1 \leq i < j \leq n$, $\omega(g_i) \leq -1$ if $1 \leq j < i \leq n$ and $\omega(g_i) = 0$ for all $1 \leq i \leq n$. 

Proof. Let $B_0 = \{b_1, \ldots, b_n\}$ be the standard basis of $F^n$ and let $C = \Psi_{B_0} \circ w(C_H)$ be a chamber of $A_0$, where $w = [a, \sigma], a = (a_1, \ldots, a_n) \in H \cap \mathbb{Z}^n, \sigma \in S_n$. Then $w$ acts on $(t_1, \ldots, t_n) \in H$ by $w(t_1, \ldots, t_n) = (t_{\sigma(1)} + a_1, \ldots, t_{\sigma(n)} + a_n)$. For $g \in N$ with $g = (g_{ij})$, $\omega(g_{\sigma(ij)}) = a_i$ (such a $g$ exists) we have, for all $f \in A_0$,

$$(gf)(b_i) = f(g^{-1}b_i) = f(g_{\sigma(i)}b_{\sigma(i)}) = \omega(g_{\sigma(i)}) + f(b_{\sigma(i)}) = a_i + f(b_{\sigma(i)}).$$

Hence for all $t \in H$ we have $\Psi_{B_0}(t) = \Psi_{B_0}(w(t))$, so $C = wC_0$. Thus $N$ acts transitively on the set of chambers of the fundamental apartment $A_0$, so $SL_n(F)$ acts transitively on the set of chambers of $T_M$.

Suppose $g \in SL_n(F)$ with $gf \in C_0$ for all $f \in C_0$, write $g^{-1} = (g_{ij})$ and let $f \in C_0$. For all $1 \leq i \leq n$ there is an $1 \leq i' \leq n$ with

$$(gf)(b_i) = f(g^{-1}b_i) = \max_{1 \leq j \leq n} \{v(b_{j}, g^{-1}b_i) + f(b_j)\} - v(b_0) = \max_{1 \leq j \leq n} \{\omega(g_{ji}) + f(b_j)\} = \omega(x_{i'i}) + f(b_{i'}).$$

Since $f \in C_0$, we have

$$f(b_1) \leq \cdots \leq f(b_n) \leq f(b_1) + 1$$

and hence $|f(b_i) - f(b_j)| < 1$ for all $1 \leq i, j \leq n$. Since $gf \in C_0 \subset A_0$ we have

$$(gf)(b_1) \leq \cdots \leq (gf)(b_n) \leq (gf)(b_1) + 1$$

and $\sum_{i=1}^n (gf)(b_i) = 0$. Thus

$$\omega(g_{1'i}) + f(b_{i'}) \leq \cdots \leq \omega(g_{n'n}) + f(b_{n'}) \leq \omega(g_{1'i}) + f(b_{i'}) + 1$$

and $\sum_{i=1}^n (\omega(g_{ij}) + f(b_{ij})) = 0$. Since $\omega(F^*) \leq \mathbb{Z}$ and $|f(b_i) - f(b_j)| < 1$ this implies that $i' = i$ and $\omega(g_{ji}) = 0$ for all $1 \leq i \leq n$. Thus $(b_i, b_j, b_k, b_{ij}, b_{ik}, b_{jk})$ for all $1 \leq i < j \leq n$. For $1 \leq j < i$, since $|f(b_i) - f(b_j)| < 1$, we have $\omega(g_{ji}) = 0$. For $i < j \leq n$, since $f(b_{ij}) \leq f(b_i)$, we have $\omega(g_{ji}) \leq -1$. Thus $g \in B$.

Conversely, if $g \in B, g^{-1} = (g_{ij}), f \in C_0$ and $1 \leq i \leq n$ then

$$(gf)(b_i) = \max_{1 \leq j \leq n} \{\omega(g_{ji}) + f(b_j)\} = \max_{1 \leq j \leq i} \{\omega(g_{ji}) + f(b_j)\}$$

since $|f(b_i) - f(b_j)| < 1$ for $1 \leq i, j \leq n$ and $\omega(g_{ji}) \leq -1$ for $i < j \leq n$. Also, $\max_{1 \leq j \leq i} \{\omega(g_{ji}) + f(b_j)\} = f(b_i)$, since $\omega(g_{ii}) = 0 \geq \omega(g_{ji})$ and $f(b_{ij}) \leq f(b_i)$ for $1 \leq j < i$. Thus $gf = f$ for every $g \in B$ and $f \in C_0$, and hence $B$ is the stabilizer of the fundamental chamber $C_0$.

It follows that $SL_n(F)$ acts strongly transitively on the set of chambers and apartments of $T_M$, with the same stabilizers $B, N$ and $T$ as the affine building for $SL_n(F)$. Hence there is a natural bijection from the set of chambers of $T_M$ to the set of chambers of the affine building for $SL_n(F)$, given by $gC_0 \mapsto gB$ (compare with the proof of Proposition 3.6).

Suppose that the distinct chambers $C, C' \in T_M$ are adjacent. Since the action of $SL_n(F)$ on $T_M$ preserves the type of vertices, and maps chambers to chambers one-to-one, it follows that $gC$ and $gC'$ are distinct adjacent chambers for all $g \in SL_n(F)$, so the action of $SL_n(F)$ on $T_M$ preserves adjacency.

We can thus define a bijection from the set of simplices of $T_M$ to the set of simplices of the affine building for $SL_n(F)$ as follows. For any simplex $D \in T_M$, let $D_\mathfrak{P} \subset SL_n(F)$ be the set of elements of $SL_n(F)$ with $D \subset gC_0$ for all $g \in D_\mathfrak{P}$. This is clearly a double coset $gB \cdot S' \cdot B$ of $SL_n(F)$ for some $g \in SL_n(F)$ and $S' \subseteq S$. Now map $D$ to $gB \cdot S' \cdot B$. This bijection preserves incidence, and is hence a simplicial isomorphism.

Since $T_M$ is clearly a geometric realization of the simplicial complex defined by the simplices of $T_M$, we have shown the following fact.

**Theorem 5.21.** Suppose that $F$ is a discretely valued field with valuation $\omega$, $E = F^n \setminus \{0\}$, $v = \omega \circ \det$, and $M = M_n(F) = (E, v)_n$. Then the tight span $T_M$ of $M$ is a geometric realization of the affine building for the group $SL_n(F)$. 
Chapter 6

Norms and the tight span

Throughout this chapter, we let $F$ be a field with a discrete valuation, and we follow the notation used in the appendix. We will now look at decomposable norms on the vector space $F^n$, as defined in [2], and prove that the elements of the tight span of $M_n(F)$ are precisely the set of decomposable norms on $F^n$ with volume zero. The definitions and results of Section 6.1 are taken from [2].

6.1 Decomposable norms

Definition 6.1. A norm on $F^n$ is a map $\alpha : F^n \to \mathbb{R} \cup \{-\infty\}$ such that, for all $k \in F$ and $x, y \in F^n$,

(N1) $\alpha(kx) = \omega(k) + \alpha(x)$,

(N2) $\alpha(x + y) \leq \max \{\alpha(x), \alpha(y)\}$ and

(N3) $\alpha(x) = -\infty$ if and only if $x = 0$.

Remark 6.2. Note that, as in the case of discrete valuations, if $\alpha(x) \neq \alpha(y)$ then $\alpha(x + y)$ is the maximum of $\alpha(x)$ and $\alpha(y)$.

Lemma 6.3. Suppose $x_1, \ldots, x_m \in F^n$ with $\alpha(x_i) \neq \alpha(x_j) \pmod{1}$ for all $i \neq j, 1 \leq i, j \leq m$. Then the vectors $x_1, \ldots, x_m$ are linearly independent. In particular, there is an integer $r$ with $1 \leq r \leq n$, and real numbers $c_1, \ldots, c_r \in [0, 1)$ so that for any $x \in F^n$ there is some $1 \leq i \leq r$ with $\alpha(x) \equiv c_i \pmod{1}$.

Proof. Suppose $x_1, \ldots, x_m \in F^n$ with $\alpha(x_i) \neq \alpha(x_j) \pmod{1}$ for all $i \neq j, 1 \leq i, j \leq m$. Then $\alpha(kx_i) \neq \alpha(x_j)$ for all $k, l \in F^*$ and $i \neq j, 1 \leq i, j \leq m$. Thus, for all $k_1, \ldots, k_m \in F^*$

$$\alpha(k_1x_1 + \cdots + k_mx_m) = \max \{\omega(k_1) + \alpha(x_1), \ldots, \omega(k_m) + \alpha(x_m)\}.$$ 

Now suppose that $x_1, \ldots, x_m$ are linearly dependent. Write $x_i = k_1x_1 + \cdots + k_mx_m$ for some $k_i \in F, 1 \leq i \leq n$. Then

$$\alpha(x_i) = \alpha(k_2x_2 + \cdots + k_mx_m) = \max \{\omega(k_2) + \alpha(x_2), \ldots, \omega(k_m) + \alpha(x_m)\}$$

which contradicts our assumption that $\alpha(x_1) \neq \alpha(x_i) \pmod{1}$ for all $2 \leq i \leq m$. Hence the vectors $x_1, \ldots, x_m$ must be linearly independent.

For the second assertion, let $x_1, \ldots, x_r$ be a maximal set with $\alpha(x_i) \neq \alpha(x_j) \pmod{1}$ for all $i \neq j$. Every such set must have the same number of elements. Also, since this set is linearly independent we must have $r \leq n$, and by multiplying by a suitable element of $F$ we may assume $\alpha(x_i) \in [0, 1)$. Set $c_i = \alpha(x_i)$. The result follows immediately.

The integer $r$ in Lemma 6.3 is called the rank of $\alpha$. Hence $\alpha(F^n \setminus \{0\})$ consists of $r$ classes mod 1. In particular, $\alpha(F^n \setminus \{0\})$ is discrete in $\mathbb{R}$. 

49
Given any basis $B = \{b_1, \ldots, b_n\}$ of $F^n$ we define a norm $\alpha_B$ by
\[
\alpha_B(k_1b_1 + \cdots + k_nb_n) = \max \{\omega(k_1), \ldots, \omega(k_n)\}.
\]
To see that this is indeed a norm, let $x = \sum_{i=1}^nk_ib_i, y = \sum_{i=1}^nl_ib_i \in F^n, k \in F$. Then
\[
\alpha_B(kx) = \alpha_B \left(k \sum_{i=1}^nk_ib_i\right) = \alpha_B \left( \sum_{i=1}^nk_i b_i\right) = \max \{\omega(k_i)\} = \omega(k) + \max_{1 \leq i \leq n} \{\omega(k_i)\} = \omega(k) + \alpha(x),
\]
and so $\alpha_B$ satisfies (N1). Also,
\[
\alpha_B(x+y) = \alpha_B \left( \sum_{i=1}^nk_ib_i + \sum_{i=1}^nl_ib_i\right) = \alpha_B \left( \sum_{i=1}^n(k_i + l_i)b_i\right) = \max_{1 \leq i \leq n} \{\omega(k_i) + \omega(l_i)\}
\leq \max_{1 \leq i \leq n} \{\max \{\omega(k_i), \omega(l_i)\}\} = \max \left\{\max_{1 \leq i \leq n} \{\omega(k_i)\}, \max_{1 \leq i \leq n} \{\omega(l_i)\}\right\} = \max \{\alpha_B(x), \alpha_B(y)\},
\]
and so (N2) is satisfied as well. Finally, $\alpha_B(x) = -\infty$ if and only if $k_1 = \cdots = k_n = 0$, that is, $x = 0$, and hence $\alpha_B$ satisfies (N3).

Note that, since $\omega(0) = -\infty$, $\alpha_B$ satisfies $\alpha_B(kb_i) = \omega(k)$ for all $k \in F$ and $1 \leq i \leq n$. Hence
\[
\alpha_B(k_1b_1 + \cdots + k_nb_n) = \max \{\alpha_B(k_1b_1), \ldots, \alpha_B(k_nb_n)\}.
\]

This motivates the following definition.

**Definition 6.4.** Let $\alpha$ be a norm on $F^n$. We say that $\alpha$ **decomposes** with respect to a basis $B = \{b_1, \ldots, b_n\}$ of $F^n$ if
\[
\alpha(k_1b_1 + \cdots + k_nb_n) = \max \{\omega(k_1) + \alpha(b_1), \ldots, \omega(k_n) + \alpha(b_n)\}
\]
for all $k_1, \ldots, k_n \in F^n$. In addition, we say that $\alpha$ is **decomposable** if such a basis $B$ exists.

A decomposable norm $\alpha$ on $F^n$ naturally defines a lattice $\mathcal{X}_\alpha$ in $F^n$, given by
\[
\mathcal{X}_\alpha = \{x \in F^n \mid \alpha(x) \leq 0\}.
\]
If $B = \{b_1, \ldots, b_n\}$ is a basis of $F^n$ such that $\alpha$ decomposes with respect to $B$, then $\mathcal{X}_\alpha$ is the lattice spanned by $\{g_1b_1, \ldots, g_nb_n\}$, where $g_i \in F^*, 1 \leq i \leq n$ are elements with $-1 < \alpha(g_ib_i) \leq 0$.

**Lemma 6.5.** Let $\alpha$ be a decomposable norm on $F^n$ with rank $r$, and let $c_1, \ldots, c_r \in [0, 1], c_1 \cdots < c_r$, represent the equivalence classes of $\alpha(F^n \setminus \{0\})$ (mod 1). Then there are integers $n_1, \ldots, n_r, 1 \leq n_i \leq r$, with $n_1 + \cdots + n_r = n$, so that for every basis $B = \{b_1, \ldots, b_n\}$ of $F^n$ the number of basis vectors $b_i$ satisfying $\alpha(b_i) \equiv c_j$ (mod 1) is $n_j$.

**Proof.** Let $B = \{b_1, \ldots, b_n\}$ and $B' = \{b'_1, \ldots, b'_n\}$ be two bases of $F^n$. Multiplying by elements of $F^*$ if necessary, we may assume $0 \leq \alpha(b_1), \ldots, \alpha(b_n) < 1$ and $0 \leq \alpha(b'_1), \ldots, \alpha(b'_n) < 1$. Let $I \subset \{1, \ldots, n\}$ be the maximal set with $\alpha(b_i) = c_1$ for all $i \in I$, and let $J \subset \{1, \ldots, n\}$ be the maximal set with $\alpha(b'_i) = c_1$ for all $i \in J$. We want to show that $|I| = |J|$. Assume for a contradiction that $|I| < |J|$. Now, write every $b_i$ as a linear combination of the elements of $B'$:
\[
b_i = \sum_{j=1}^n k_{ij}b'_j.
\]
Then,
\[
\alpha(b_i) = \max_{1 \leq j \leq n} \{\omega(k_{ij}) + \alpha(b'_j)\},
\]
and since $\alpha(b_i) = \alpha(b'_j)$ if $i \in I, j \in J$ and $\alpha(b_i) \leq \alpha(b'_j)$ if $i \in I, j \notin J$, it follows that $\omega(k_{ij}) = 0$ for all $j \in J$ and $k_{ij} = 0$ for all $j \notin J$. Hence for all $i \in I$ the vector $b_i$ is in the subspace spanned by the set $\{b'_j \mid j \in J\}$. But $\{b_i \mid i \in I\}$ is a set of $|I|$ linearly independent vectors, and the subspace spanned by the set $\{b'_j \mid j \in J\}$ is of dimension $|J| < |I|$, a contradiction. Hence $|I| = |J|$. Put $n_1 = |I|$.

Now, let $I'$ be the maximal set with $\alpha(b_i) = c_2$ for all $i \in I'$ and let $J'$ be the maximal set such that $\alpha(b_i) = c_2$ for all $i \in J'$. Using the same dimension argument, we see that we must have $n_1 + |I'| = n_1 + |J'|$. Continuing in this manner, we find the integers $n_1, \ldots, n_r$ as required. □
6.1. DECOMPOSABLE NORMS

The sequence \((n_1, \ldots, n_r)\) given by the last lemma is called the type of \(\alpha\). It turns out that a decomposable norm decomposes with respect to a large number of bases, which is essentially the result given in Corollary 6.9. But to prove this, we need the following results.

**Lemma 6.6.** Given any two bases \(B = \{b_1, \ldots, b_n\}\) and \(B' = \{b'_1, \ldots, b'_n\}\) of \(F^n\), there is a \(c \in \mathbb{R}\) such that \(\alpha_B(x) \leq \alpha_{B'}(x) + c\) for all \(x \in F^n\).

**Proof.** Suppose \(b'_i = \sum_{j=1}^{n} l_{ij} b_j\), and let \(x = \sum_{i=1}^{n} k_i b'_i \in F^n\) be any element of \(F^n\). Then

\[
\alpha_B(x) = \alpha_B \left( \sum_{i=1}^{n} k_i b'_i \right) = \alpha_B \left( \sum_{i=1}^{n} k_i \left( \sum_{j=1}^{n} l_{ij} b_i \right) \right) = \alpha_B \left( \sum_{j=1}^{n} \left( \sum_{i=1}^{n} k_i l_{ij} \right) b_j \right) = \max_{1 \leq i \leq n} \left\{ \omega \left( \sum_{j=1}^{n} k_i l_{ij} \right) \right\} \leq \max_{1 \leq i \leq n} \left\{ \max_{1 \leq j \leq n} \{ \omega(k_i) + \omega(l_{ij}) \} \right\} = \max_{1 \leq i,j \leq n} \{ \omega(k_i) + \omega(l_{ij}) \},
\]

Set \(c = \max_{1 \leq i,j \leq n} \{ \omega(l_{ij}) \}\) and the result follows. 

**Corollary 6.7.** Let \(\alpha\) be a decomposable norm on \(F^n\). Then for all bases \(B = \{b_1, \ldots, b_n\}\) there is a \(c \in \mathbb{R}\) such that \(\alpha(x) \geq \alpha_B(x) + c\) for all \(x \in F^n\).

**Proof.** Let \(B' = \{b'_1, \ldots, b'_n\}\) be a basis of \(F^n\) such that \(\alpha\) decomposes with respect to \(B'\). Multiplying by elements of \(F^n\) if necessary, we may assume that \(0 \leq \alpha(b'_i) < 1\) for all \(1 \leq i \leq n\). Then \(\alpha(x) \geq \alpha_B(x)\) for all \(x \in F^n\). The result now follows by Lemma 6.6.

**Lemma 6.8.** Let \(\alpha\) be a decomposable norm on \(F^n\), let \(b_1, \ldots, b_n \in F^n\), \(1 \leq l < n\) be a set of linearly independent vectors and let \(X\) be the subspace of \(F^n\) spanned by the vectors \(b_1, \ldots, b_l\). If \(\alpha\) restricted to \(X\) decomposes with respect to the basis \(\{b_1, \ldots, b_l\}\) of \(X\), then there is an element \(b_{l+1} \in F^n \setminus X\) such that \(\alpha\) restricted to the subspace \(Y\) of \(F^n\) spanned by the vectors \(b_1, \ldots, b_{l+1}\) decomposes with respect to the basis \(\{b_1, \ldots, b_{l+1}\}\) of \(Y\).

**Proof.** Let \(b_{l+1}, \ldots, b_n \in F^n\) be elements such that \(B = \{b_1, \ldots, b_n\}\) is a basis of \(F^n\). Let \(x_0 \in F^n \setminus X\). Then \(x_0 = \sum_{i=1}^{l} k_i b_i\) for some \(k_i \in F\), where \(k_i \neq 0\) for some \(i \leq n\) as \(x_0 \notin X\). Since \(\alpha_B(x_0 + x) \geq \max_{1 \leq i \leq n} \{ \omega(k_i) \}\) for all \(x \in X\) it follows that inf \(\{ \alpha_B(x_0 + x) \mid x \in X\} > -\infty\). By Corollary 6.7 there is a \(c \in \mathbb{R}\) such that \(\alpha(x) \geq \alpha_B(x) + c\) for all \(x \in F^n\). Hence

\[
\inf \{ \alpha(x_0 + x) \mid x \in X\} \geq \inf \{ \alpha_B(x_0 + x) \mid x \in X\} + c > -\infty.
\]

Since the set \(\alpha(x_0 + X)\) is discrete in \(\mathbb{R}\), it follows that there is an element \(b'_{l+1} \in x_0 + X\) such that \(\alpha(b'_{l+1}) = \inf \{ \alpha(x_0 + x) \mid x \in X\}\). Thus

\[
\alpha(b'_{l+1}) \leq \alpha(b'_{l+1} + x)
\]

for all \(x \in X\). Let \(x \in X\). If \(\alpha(b'_{l+1}) < \alpha(x)\) then by Remark 6.2 \(\alpha(b'_{l+1} + x) = \max \{ \alpha(b'_{l+1}), \alpha(x) \}\), so assume \(\alpha(b'_{l+1}) \geq \alpha(x)\). Then

\[
\alpha(b'_{l+1}) = \max \{ \alpha(b'_{l+1}), \alpha(x) \} \geq \alpha(b'_{l+1} + x) \geq \alpha(b'_{l+1}),
\]

and hence \(\alpha(b'_{l+1} + x) = \max \{ \alpha(b'_{l+1}), \alpha(x) \}\) for all \(x \in X\). Now, let \(k \in F\). Then

\[
\alpha(kb'_{l+1} + x) = \omega(k) + \alpha(b'_{l+1} + k^{-1} x) = \omega(k) + \max \{ \alpha(b'_{l+1}, - \omega(k) + \alpha(x) \}
\]

for all \(x \in X\). Hence \(\alpha\) restricted to the subspace of \(F^n\) spanned by the vectors \(b_1, \ldots, b_l, b'_{l+1}\) decomposes with respect to the basis \(\{b_1, \ldots, b_l, b'_{l+1}\}\), and we are done.

**Corollary 6.9.** If \(\alpha\) is a decomposable norm on \(F^n\), then any non-zero element \(b_1 \in F^n\) can be extended to a basis of \(F^n\) such that \(\alpha\) decomposes with respect to this basis.

**Proof.** Any decomposable norm \(\alpha\) restricted to the subspace spanned by a vector \(b_1\) decomposes on the subspace spanned by \(b_1\). Hence applying Lemma 6.8 \((n - 1)\) times, the result follows.
In the Appendix (Proposition A.8 (iv)) we show that if \( g, g' \in GL_n(F) \) with \( gO^n = g' O^n \) (where \( O^n \) is the lattice spanned by the standard lattice), then \( \omega \circ \det(g) = \omega \circ \det(g') \). In light of this fact, we define the volume of a lattice \( X = gO^n \) to be
\[
\text{vol}(X) = \omega \circ \det(g).
\]
If a basis \( B \) spans \( gO^n \), we define \( \text{vol}(B) = \text{vol}(gO^n) \). Note that if \( g \in GL_n(F) \) and \( B \) is a basis of \( F^n \), then \( \text{vol}(gB) = \omega \circ \det(g) + \text{vol}(B) \).

**Proposition 6.10.** Let \( \alpha \) be a decomposable norm on \( F^n \). Then, if \( \alpha \) decomposes with respect to the bases \( B = \{b_1, \ldots, b_n\} \) and \( B' = \{b'_1, \ldots, b'_n\} \), then
\[
\text{vol}(B) - \sum_{i=1}^{n} \alpha(b_i) = \text{vol}(B') - \sum_{i=1}^{n} \alpha(b'_i).
\]

**Proof.** Suppose \( \alpha \) decomposes with respect to the bases \( B = \{b_1, \ldots, b_n\} \) and \( B' = \{b'_1, \ldots, b'_n\} \). Suppose \( k_1, \ldots, k_n, k'_1, \ldots, k'_n \in F^* \) with (after possibly reordering)
\[
-1 < \alpha(k_1b_1) \leq \cdots \leq \alpha(k_nb_n) \leq 0
\]
and
\[
-1 < \alpha(k'_1b'_1) \leq \cdots \leq \alpha(k'_nb'_n) \leq 0.
\]
Both bases \( \{k_1b_1, \ldots, k_nb_n\} \) and \( \{k'_1b'_1, \ldots, k'_nb'_n\} \) span the lattice \( X_\alpha \). Now, let \( g, g' GL_n(F) \) be elements with \( X_\alpha = g \{b_1, \ldots, b_n\} \) and \( X'_\alpha = g' \{b'_1, \ldots, b'_n\} \). Clearly, \( \det(g) = k_1 \cdots k_n \) and \( \det(g') = k'_1 \cdots k'_n \) so
\[
\text{vol}(B) + \sum_{i=1}^{n} \omega(k_i) = \text{vol}(X_\alpha) = \text{vol}(B') + \sum_{i=1}^{n} \omega(k'_i).
\]
Also, by Lemma 6.5, if \( \alpha \) is of rank \( r \) and type \( (n_1, \ldots, n_r) \) we have
\[
\sum_{i=1}^{n} \omega(k_i) + \sum_{i=1}^{n} \alpha(b_i) = \sum_{i=1}^{n} \alpha(k_i b_i) = \sum_{i=1}^{n} n_i c_i = \sum_{i=1}^{n} \alpha(k'_i b'_i) = \sum_{i=1}^{n} \omega(k'_i) + \sum_{i=1}^{n} \alpha(b'_i).
\]
Hence
\[
\text{vol}(B) - \text{vol}(B') = \sum_{i=1}^{n} \omega(k'_i) - \sum_{i=1}^{n} \omega(k_i) = \sum_{i=1}^{n} \alpha(b'_i) - \sum_{i=1}^{n} \alpha(b_i),
\]
and so
\[
\text{vol}(B) - \sum_{i=1}^{n} \alpha(b_i) = \text{vol}(B') - \sum_{i=1}^{n} \alpha(b'_i).
\]
\[\square\]

Suppose \( \alpha \) is a decomposable norm on \( F^n \) so that \( \alpha \) decomposes with respect to a basis \( B = \{b_1, \ldots, b_n\} \) of \( F^n \). We define the volume of \( \alpha \) to be
\[
\text{vol}(\alpha) = \text{vol}(B) - \sum_{i=1}^{n} \alpha(b_i).
\]
By the previous lemma, the volume of a decomposable norm is well-defined.

### 6.2 Relating decomposable norms and the tight span

We have now come to the point where we will show the main result of this thesis. The decomposable norms on \( F^n \) and the elements of the tight span of \( M_n(F) \) have some similarities. For example, they are both functions from \( F^n \) to \( \mathbb{R} \cup \{-\infty\} \), and by Lemma 5.16 the elements of the tight span satisfy (N1) and by definition they satisfy (N3).

Recall that the valuation \( v : (F^n \setminus \{0\})^n \rightarrow \mathbb{Z} \cup \{-\infty\} \) is defined by
\[
v(e_1, \ldots, e_n) = \omega \circ \det(e_1, \ldots, e_n) = \omega \begin{vmatrix} e_{11} & \cdots & e_{n1} \\ \vdots & \ddots & \vdots \\ e_{1n} & \cdots & e_{nn} \end{vmatrix}.
\]
We start with a rather simple lemma.
Lemma 6.11. Let \( B = \{b_1, \ldots, b_n\} \) be any basis of \( F^n \) (and equivalently of \( M_n(F) \)). Then \( \text{vol}(B) = v(B) \).

Proof. Let \( \mathcal{X} \) be the lattice spanned by \( B \) and let \( g \in GL_n(F) \) be defined by the matrix
\[
g = \begin{bmatrix}
    b_{11} & \cdots & b_{n1} \\
    \vdots & \ddots & \vdots \\
    b_{1n} & \cdots & b_{nn}
\end{bmatrix},
\]
where \( b_i = (b_{i1}, \ldots, b_{in}), 1 \leq i \leq n \). Then \( gO^n = \mathcal{X} \) and \( \det(g) = \det(B) \), so
\[
\text{vol}(B) = \omega \circ \det(g) = \omega \circ \det(B) = v(B).
\]

\[\Box\]

Proposition 6.12. Every element of \( T_M \) is a norm on \( F^n \).

Proof. In view of the above remarks we only have to show that every \( f \in T_M \) satisfies (N2). Let \( e_1, e_2 \in F^n \). By Proposition 5.5 there is some basis \( B = \{b_1, \ldots, b_n\} \) and some \( t = (t_1, \ldots, t_n) \in H \) with \( f = \Psi_B(t) \), where \( \Psi_B \) is given in Proposition 5.5. Hence
\[
f(e_1 + e_2) = \max_{1 \leq i \leq n} \left\{ v(e_1 + e_2, \hat{b}_i) + t_i \right\} - \left( \frac{n-1}{n} \right) v(B)
\leq \max_{1 \leq i \leq n} \left\{ \max \left\{ v(e_1, \hat{b}_i), v(e_2, \hat{b}_i) \right\} + t_i \right\} - \left( \frac{n-1}{n} \right) v(B)
= \max \left\{ \max_{1 \leq i \leq n} \left\{ v(e_1, \hat{b}_i), v(e_2, \hat{b}_i) \right\} + t_i \right\} - \left( \frac{n-1}{n} \right) v(B) = \max \left\{ f(e_1), f(e_2) \right\}.
\]

\[\Box\]

Lemma 6.13. Let \( f \in T_M \). If \( B = \{b_1, \ldots, b_n\} \) is a basis of \( M_n(F) \) with \( f \in A_B \), then \( f \) decomposes with respect to \( B \).

Proof. Let \( f \) and \( B \) be as in the lemma. Since \( B \) is a basis of \( F^n \), every \( e \in E \) can be written as \( e = \sum_{i=1}^n k_i b_i \) for some \( k_i \in F \). By Proposition 5.5 there is some \( t = (t_1, \ldots, t_n) \in H \) with \( f = \Psi_B(t) \). Hence
\[
f(e) = f \left( \sum_{i=1}^n k_i b_i \right) = \max_{1 \leq i \leq n} \left\{ v \left( \sum_{i=1}^n k_i b_i, \hat{b}_i \right) + t_i \right\} - \left( \frac{n-1}{n} \right) v(B)
= \max_{1 \leq i \leq n} \left\{ \omega \left( \det \left( \sum_{i=1}^n k_i b_i, \hat{b}_j \right) \right) \right\} - \left( \frac{n-1}{n} \right) v(B)
= \max_{1 \leq i \leq n} \left\{ \omega \left( \sum_{i=1}^n k_i \det (b_i, \hat{b}_j) \right) + t_j \right\} - \left( \frac{n-1}{n} \right) v(B),
\]
where the last equality follows from the multi-linearity of the determinant. Now, since \( \det(b_i, \hat{b}_j) \neq -\infty \) if and only if \( i \neq j, 1 \leq i, j \leq n \), this can be reduced to
\[
f(e) = \max_{1 \leq i \leq n} \left\{ \omega (\det(k_i b_j, \hat{b}_j)) + t_j \right\} - \left( \frac{n-1}{n} \right) v(B)
= \max_{1 \leq j \leq n} \left\{ v(k_j b_j, \hat{b}_j) + t_j \right\} - \left( \frac{n-1}{n} \right) v(B).
\]
Since \( v(k_i b_i, \hat{b}_j) = -\infty \) if \( i \neq j \), we can write \( v(k_j b_j, \hat{b}_j) = \max_{1 \leq i \leq n} \left\{ v(k_i b_i, \hat{b}_j) \right\} \), and hence
\[
f(e) = \max_{1 \leq i \leq n} \left\{ \max_{1 \leq i \leq n} \left\{ v(k_i b_i, \hat{b}_j) + t_j \right\} \right\} - \left( \frac{n-1}{n} \right) v(B)
= \max_{1 \leq i \leq n} \left\{ \max_{1 \leq i \leq n} \left\{ v(k_i b_i, \hat{b}_j) + t_j \right\} \right\} - \left( \frac{n-1}{n} \right) v(B) = \max_{1 \leq i \leq n} \left\{ f(k_i b_i) \right\}.
\]
Hence \( f \) decomposes with respect to the basis \( B \).
Now suppose \( f \in T_M \). Let \( B = \{b_1, \ldots, b_n\} \) be a basis of \( M_n(F) \) with \( f \in A_B \). By Lemma 5.7 we have
\[
v(B) - \sum_{i=1}^{n} f(b_i) = 0,
\]
and by Lemma 6.13 \( f \) decomposes with respect to the basis \( B \). Hence
\[
\text{vol}(f) = \text{vol}(B) - \sum_{i=1}^{n} f(b_i) = \text{vol}(B) - \sum_{i=1}^{n} f(b_i) = 0.
\]
Thus if \( f \in T_M \) then \( f \) is a decomposable norm on \( F^n \) with volume zero.

We now conclude with our main result.

**Theorem 6.14.** The decomposable norms on \( F^n \) of volume zero are precisely the elements of the tight span of \( M_n(F) \).

**Proof.** By the discussion above, we only need to show that every norm on \( F^n \) with volume zero is in the tight span of \( M_n(F) \). Suppose \( \alpha \) is a norm of volume zero, and suppose \( \alpha \) decomposes with respect to a basis \( B = \{b_1, \ldots, b_n\} \). To show \( \alpha \in T_M \), we need to show
\[
\alpha(e) = \max_{e_2, \ldots, e_n \in E} \left\{ v(e, e_2, \ldots, e_n) - \sum_{i=2}^{n} f(e_i) \right\}
\]
for all \( e \in E \). Given \( e \in E \), by Corollary 6.9 there are elements \( e_2, \ldots, e_n \in E \) such that \( \alpha \) decomposes with respect to the basis \( \{e, e_2, \ldots, e_n\} \). Since \( \alpha \) has volume zero it follows that \( \alpha(e) = v(e, e_2, \ldots, e_n) - \sum_{i=2}^{n} \alpha(e_i) \). Thus we have to show \( \alpha(e_1) \leq v(e_1, \ldots, e_n) - \sum_{i=2}^{n} \alpha(e_i) \), or equivalently, \( \sum_{i=1}^{n} \alpha(e_i) \leq v(e_1, \ldots, e_n) \) for all \( e_1, \ldots, e_n \in E \).

Let \( e_1, \ldots, e_n \in E \) be any elements of \( E \). Since \( B \) is a basis of \( F^n \) we can write \( e_i = \sum_{j=1}^{n} k_{ij} b_j \) for some \( k_{ij} \in F^* \), \( 1 \leq i \leq n \). Then
\[
v(e_1, \ldots, e_n) = v \left( \sum_{j=1}^{n} k_{1j} b_j, \ldots, \sum_{j=1}^{n} k_{nj} b_j \right)
\]
\[
\leq \max_{\rho \in R} \left\{ v(k_{1\rho(1)} b_{\rho(1)}, \ldots, k_{n\rho(n)} b_{\rho(n)}) \right\}
\]
\[
= \max_{\rho \in R} \left\{ \sum_{i=1}^{n} \omega(k_{i\rho(i)}) + v(b_{\rho(1)}, \ldots, b_{\rho(n)}) \right\}
\]
where \( R \) is the set of functions from \( \{1, \ldots, n\} \) to itself. Now, since \( v(b_{\rho(1)}, \ldots, b_{\rho(n)}) \neq -\infty \) if and only if \( \rho \) is a bijection, and noting that \( \sum_{i=1}^{n} \alpha(b_i) = v(B) \), we can continue by writing
\[
v(e_1, \ldots, e_n) \leq \max_{\sigma \in S_n} \left\{ \sum_{i=1}^{n} \omega(k_{i\sigma(i)}) + v(b_1, \ldots, b_n) \right\}
\]
\[
= \max_{\sigma \in S_n} \left\{ \sum_{i=1}^{n} \left( \omega(k_{i\sigma(i)}) + \alpha(b_{\sigma(i)}) \right) \right\}
\]
\[
\leq \sum_{i=1}^{n} \max_{\sigma \in S_n} \{ \alpha(k_{i\sigma(i)} b_{\sigma(i)}) \} = \sum_{1 \leq i \leq n} \max_{1 \leq j \leq n} \{ \alpha(k_{ij} b_j) \}
\]
\[
= \sum_{i=1}^{n} \alpha(e_i).
\]
This completes the proof. \( \square \)
Appendix A

Discretely valued fields

A.1 Discrete valuations

Definition A.1. A discrete valuation on a field $F$ is a surjective homomorphism $\omega$ from $F^*$, the multiplicative group of $F$, to the group $\mathbb{Z}$ under addition satisfying the inequality

$$\omega(x + y) \leq \max\{\omega(x), \omega(y)\}$$

for all $x, y \in F^*$ such that $x + y \neq 0$.

Remark A.2. Note that we will make no distinction between the additive and multiplicative identities of $F$ and 0 and 1 of $\mathbb{Z}$, but denote these by 0 and 1 respectively. Also note that the homomorphism $\omega$ maps the multiplicative group of $F$ to the additive group $\mathbb{Z}$, so the homomorphism satisfies $\omega(xy) = \omega(x) + \omega(y)$, which might be confusing at first sight.

Any discrete valuation can be extended to all of $F$ by setting $\omega(0) = -\infty$, in which case the above inequality holds for all $x, y \in F$. Since $\omega$ is a homomorphism, 1 is the identity in $F^*$ and 0 is the identity in $\mathbb{Z}$ we necessarily have $\omega(1) = 0$. Furthermore, we have $0 = \omega(1) = \omega((-1)(-1)) = \omega(-1) + \omega(-1)$ and so we must have $\omega(-1) = 0$. Hence, for all $x \in F$,

$$\omega(-x) = \omega((-1)x) = \omega(1) + \omega(x) = \omega(x).$$

Example A.3 Consider the field $\mathbb{Q}$, and choose a prime $p$. Then for every element $q \in \mathbb{Q}$ there is a unique $n \in \mathbb{Z}$ and elements $r_1, r_2 \in \mathbb{Z}$ with $q = p^n \frac{r_1}{r_2}$, where $p|r_1$ and $p \nmid r_2$. The map $\omega_p : \mathbb{Q} \to \mathbb{Z}$ defined by

$$\omega_p \left( p^n \frac{r_1}{r_2} \right) = -n$$

is then easily seen to be a discrete valuation on $\mathbb{Q}$.

Suppose $\omega(x) > \omega(y)$. Then

$$\omega(x) = \omega((x + y) - y) \leq \max\{\omega(x + y), \omega(y)\} \leq \max\{\omega(x), \omega(y)\} = \omega(x),$$

and hence $\omega(x + y) = \omega(x) = \max\{\omega(x), \omega(y)\}$. Thus, if $\omega(x) \neq \omega(y)$ then $\omega(x + y) = \max\{\omega(x), \omega(y)\}$.

The valuation $\omega$ induces a metric $d_\omega$ on $F$ given by

$$d_\omega(x, y) = e^{\omega(x-y)}.$$  

This is indeed a metric since it is symmetric, $d_\omega(x, y) = 0$ if and only if $x = y$, and for all elements $x, y, z \in F$ we have

$$d_\omega(x, z) = e^{\omega(x-z)} = e^{\omega((x-y)+(y-z))} \leq e^{\omega(x-y)+\omega(y-z)}$$

$$= \max\{e^{\omega(x-y)}, e^{\omega(y-z)}\} = \max\{d_\omega(x, y), d_\omega(y, z)\} \leq d_\omega(x, y) + d_\omega(y, z).$$

The metric $d_\omega$ satisfies the ultra-metric property, that is,

$$d_\omega(x, z) \leq \max\{d_\omega(x, y), d_\omega(y, z)\}$$
for all \( x, y, z \in F \).

Note that a sequence \((x_i)\) in \( F \) is a Cauchy sequence if and only if \( \lim_{i \to \infty} \omega(x_{i+1} - x_i) = -\infty \).

The metric space \((\mathbb{Q}, \omega_p)\) defined in Example A.3 is not complete. However, much in the same way as \( \mathbb{R} \) can be constructed as the completion of \( \mathbb{Q} \) with the usual metric on \( \mathbb{Q} \), we can construct the completion \( \mathbb{Q}_p \) of \( \mathbb{Q} \) with respect to the metric \( d_{\omega_p} \). We will not do this here, but the interested reader is referred to the chapter on valuation theory in [4].

**Lemma A.4.** Let \( F \) be a field with a discrete valuation \( \omega \). The set
\[
\mathcal{O} = \{ x \in F \mid \omega(x) \leq 0 \}
\]
forms a subring of \( F \).

*Proof.* There are three things that need to be checked
- \( 0 \in \mathcal{O} \),
- for all \( x, y \in \mathcal{O}, (x - y) \in \mathcal{O} \), and
- for all \( x, y \in \mathcal{O}, xy \in \mathcal{O} \).

First, \( \omega(0) = -\infty \leq 0 \), so \( 0 \in \mathcal{O} \). Second, for \( x, y \in \mathcal{O} \), we have
\[
\omega(x - y) = \omega(x + (-1)y) \leq \max\{\omega(x), \omega(-1) + \omega(y)\} = \max\{\omega(x), \omega(y)\} \leq 0,
\]
and so \( x - y \in \mathcal{O} \). Finally, if \( x, y \in \mathcal{O} \), then
\[
\omega(xy) = \omega(x) + \omega(y) \leq 0
\]
and so \( xy \in \mathcal{O} \). \( \square \)

Any ring \( \mathcal{O} \) that is derived in this way from a field with a discrete valuation is called a **discrete valuation ring**.

**Lemma A.5.** The multiplicative group \( \mathcal{O}^* \) of a discrete valuation ring \( \mathcal{O} \) is precisely the kernel of the discrete valuation \( \omega \).

*Proof.* Let \( x \) be an element of \( \mathcal{O}^* \), i.e. \( x \in \mathcal{O} \) and \( x^{-1} \in \mathcal{O} \). Then \( \omega(x) \leq 0 \) and \( \omega(x^{-1}) \leq 0 \). But \( \omega(x^{-1}) = -\omega(x) \) which can only be satisfied if \( \omega(x) = \omega(x^{-1}) = 0 \). So \( \mathcal{O}^* \subseteq \text{Ker} \omega \).

On the other hand, if \( x \in \text{Ker} \omega \) (i.e. \( \omega(x) = 0 \)), then \( x \neq 0 \), so there is an inverse \( x^{-1} \in F^* \). But \( \omega(x^{-1}) = -\omega(x) = 0 \), and so \( x, x^{-1} \in \mathcal{O}^* \). Hence \( \text{Ker} \omega \subseteq \mathcal{O}^* \). So \( \mathcal{O}^* = \text{Ker} \omega \). \( \square \)

**Lemma A.6.** Suppose \( \pi \in F \) with \( \omega(\pi) = -1 \). Then every element \( x \in F^* \) can be written in the form \( x = \pi^n u \) in a unique way, where \( n \in \mathbb{Z} \) and \( u \in \mathcal{O}^* \).

*Proof.* Suppose \( x \in F^* \), and let \( -n = \omega(x) \). Put \( u = x\pi^{-n} \). Then
\[
\omega(u) = \omega(x\pi^{-n}) = \omega(x) - n\omega(\pi) = -n + n = 0.
\]
Hence \( u \in \text{Ker} \omega = \mathcal{O}^* \). But then \( x = \pi^n u \), and by necessity the elements \( n \in \mathbb{Z} \) and \( u \in \mathcal{O}^* \) are unique. \( \square \)

If \( \pi \in F \) with \( \omega(\pi) = -1 \), then for any \( x \in \mathcal{O} \) we have \( \omega(\pi x) = -1 + \omega(x) \). Since \( \mathcal{O} \) is precisely those elements with \( \omega(x) \geq 0 \), we see that the ideal \( \pi \mathcal{O} \) is precisely the set \( \{ x \in F \mid \omega(x) \leq -1 \} \). Hence \( \pi \mathcal{O} \) consists of precisely those elements of \( \mathcal{O} \) that are not units. Any ideal of \( \mathcal{O} \) properly containing \( \pi \mathcal{O} \) must thus contain some units, and hence all of \( \mathcal{O} \), and so \( \pi \mathcal{O} \) is a maximal ideal.

**Proposition A.7.** Let \( \mathcal{O} \) be a discrete valuation ring derived from a field \( F \) with a discrete valuation \( \omega \) and let \( \pi \) be any element of \( F \) with \( \omega(\pi) = -1 \). Then \( \mathcal{O} \) is a principal ideal domain, and every non-zero ideal is generated by \( \pi^n \) for some \( n \geq 0 \).

*Proof.* Let \( I \) be an ideal of \( \mathcal{O} \), let \( n = \min \{-\omega(x) \mid x \in I \} \), and let \( x \) be such that \( \omega(x) = -n \). Then by Lemma A.6 we can write \( x = \pi^n u \) for some \( u \in \mathcal{O}^* \). Any ideal \( I \) satisfies \( Iy \subseteq I \) for all \( y \in \mathcal{O} \), and so in particular \( xu^{-1} = \pi^n uu^{-1} = \pi^n \in I \).

Also, \( \pi^n \) clearly divides every element of \( I \), since any element \( y \in F \) can be written as \( \pi^m u \) for some \( m \in \mathbb{Z} \) and \( u \in F^* \), and by the definition of \( n \) we must have \( m \geq n \) if \( y \in I \). Hence \( I = \pi^n \mathcal{O} \). \( \square \)

It follows that the quotient \( \mathcal{O}/\pi \mathcal{O} \) is a field, which we will denote by \( \kappa \). For the field \( \mathbb{Q} \) with valuation \( \omega_p \) it can be shown that \( \kappa = \mathbb{F}_p \), the finite field of order \( p \).
A.2 $O$-lattices

Consider the vector space $F^n$. An $O$-lattice (or simply a lattice) is a rank $n$ $O$-submodule of $F^n$. Given a basis $B$ of $F^n$, we denote the lattice spanned by $B$ by $[B]$. Hence the lattice $[b_1, \ldots, b_n]$ is the set $\{Ob_1 + \cdots + Ob_n\}$. The lattice spanned by the standard basis of $F^n$ is called the standard lattice of $F^n$ and is denoted $O^n$.

The group $GL_n(F)$ acts naturally on the set of lattices through the action defined by

$$g[b_1, \ldots, b_n] = [gb_1, \ldots, gb_n].$$

The action of $GL_n(F)$ is transitive on the set of lattices, since it is transitive on the set of bases of $F^n$.

Let $GL_n(O)$ denote the subgroup of $GL_n(F)$ that stabilizes the standard lattice. That is, $GL_n(O)$ is the subgroup of elements $g \in GL_n(F)$ satisfying $gO^n = O^n$. The matrix representation of an element $g$ of $GL_n(O)$ with respect to the standard basis is obviously an invertible matrix, and its entries must all be elements of $O$. Hence we conclude that the group $GL_n(O)$ is precisely the set of elements of $GL_n(F)$ whose matrix representations with respect to the standard basis are matrices having all entries in $O$. Since all elements of $GL_n(F)$ are invertible, these matrices are invertible as well. We will denote this group of matrices by $M_n^+(O)$. Note that since $det(A) \in O^*$ for all $A \in M_n^+(O)$ it follows that $\omega \circ det(A) = 0$ for all $A \in M_n^+(O)$.

Suppose that $g, g' \in GL_n(F)$ with $gO^n = g'O^n$. Then $g^{-1}g'O^n = O^n$ and so $g^{-1}g' \in GL_n(O)$, and hence $0 = \omega \circ det(g^{-1}g') = -\omega \circ det(g) + \omega \circ det(g')$ so $\omega \circ det(g) = \omega \circ det(g')$.

Now, let $GL_n(O)_B$ denote the stabilizer of the lattice spanned by the basis $B$. It should be obvious that the set of matrix representations with respect to the basis $B$ of the elements of $GL_n(O)_B$ is precisely the group $M_n^+(O)$.

Suppose that we have two lattices $[B]$ and $[B']$, and some $g \in GL_n(F)$ with $[B'] = g[B]$. Let $h$ be any element of $GL_n(O)_B$. Then, since $h[B] = [B]$,

$$ghg^{-1}[B'] = gh[B] = g[B] = [B'],$$

so $ghg^{-1} \in GL_n(O)_B$ and hence $gGL_n(O)_Bg^{-1} \subseteq GL_n(O)_B$. On the other hand, suppose that $h' \in GL_n(O)_{B'}$. Then $h'g[B] = g[B]$ so

$$g^{-1}h'g[B] = g^{-1}g[B] = [B],$$

and hence $g^{-1}h'g \in GL_n(O)_B$. Let $h = g^{-1}h'g$, so that $h' = ghg^{-1} \in gGL_n(O)_Bg^{-1}$. Hence we conclude that $GL_n(O)_{B'} = gGL_n(O)_{B}g^{-1}$. It follows that $\omega \circ det(g) = 0$ for all $g \in GL_n(O)_B$ for any basis $B$ of $F^n$. We sum up the above discussion on lattices in the following proposition.

Proposition A.8.

(i) $GL_n(F)$ acts transitively on the set of lattices.

(ii) If $[B]$ and $[B']$ are two lattices, and $g \in GL_n(F)$ with $g[B] = [B']$, then $GL_n(O)_{B'} = gGL_n(O)_{B}g^{-1}$.

(iii) An $n \times n$-matrix $A = \{a_{ij}\}$ is a matrix representation with respect to the basis $B$ of an element of $GL_n(O)_B$ if and only if $A \in M_n^+(O)$, or equivalently $\omega(a_{ij}) \leq 0$ for all $1 \leq i, j \leq n$, and $\omega(det(A)) = 0$.

(iv) For all bases $B$ of $F^n$ we have $\omega \circ det(g) = 0$ for all $g \in GL_n(O)_B$. Also, if $g, g' \in GL_n(F)$ satisfy $gO = g'O$, then $\omega \circ det(g) = \omega \circ det(g')$.

Proposition A.9. Let $L$ and $L'$ be any two $O$-lattices. Then there exists a basis $B = \{b_1, \ldots, b_n\}$ and elements $\lambda_1, \ldots, \lambda_n \in F^*$ such that $L = [b_1, \ldots, b_n]$ and $L' = [\lambda_1 b_1, \ldots, \lambda_n b_n]$.

Proof. Let $B' = \{b'_1, \ldots, b'_n\}$ be a basis with $L' = [b'_1, \ldots, b'_n]$, write $B'$ as a linear combination of $B$,

$$b'_i = a_{i1}b_1 + \cdots + a_{in}b_n, 1 \leq i \leq n,$$

and let $A = \{a_{ij}\}$ be the transition matrix from $B$ to $B'$. Now, acting on $L'$ with an element $g \in GL_n(O)_{B'}$ does not change $L'$, it only replaces the basis $B'$ with some other basis spanning $L'$. This change of basis corresponds to multiplying the transition matrix $A$ on the right with the matrix for $g$ with respect to the basis $B'$. Conversely, multiplying $A$ on the right with an element
APPENDIX A. DISCRETELY VALUATED FIELDS

of $M^*_n(O)$ corresponds to a linear transformation of the basis $B'$ by an element $g \in GL_n(O)_{B'}$, i.e. $gL' = L'$.

Now, let $a_{ij}$ be an entry of $A$ with maximal valuation. For each $k \neq i, 1 \leq k \leq n$, let $J_k$ be the matrix with all entries zero except the $(k, i)$'th entry, whose value is $-a_{kj}a_{ij}^{-1}$. Since $\omega(a_{ij}) \geq \omega(a_{kj})$ by the maximality of $a_{ij}$, it follows that

$$\omega(-a_{kj}a_{ij}^{-1}) = \omega(a_{kj}) - \omega(a_{ij}) \leq 0,$$

and hence $(I - J_k) \in M^*_n(O)$. Also, when multiplying $A$ by $(I - J_k)$ on the right, the $(k, j)$'th entry is canceled out. Using exactly the same procedure, but successively multiplying $A$ on the left instead of the on the right, one can cancel out all entries but $a_{ij}$ in the $i$'th column.

Now repeat the same procedure for the next element with maximal valuation, and so on. This reduces $A$ to a monomial matrix $A'$ by only multiplying on the right and left by matrices from $M^*_n(O)$, which corresponds to a change of the bases $B$ and $B'$ fixing the lattices $L$ and $L'$. Hence, by possibly rearranging the basis elements, we have transformed the basis $B$ to a basis $B'' = \{b''_1, \ldots, b''_n\}$ with $[B] = [B''] = L$ and the basis $B'$ to a basis $B''' = \{b'''_1, \ldots, b'''_n\}$ with $[B'] = [B'''] = L'$ and

$$b'''_i = \lambda_i b''_i, 1 \leq i \leq n,$$

where the $\lambda_i$'s are the nonzero elements of the monomial matrix $A'$.

Now define an equivalence relation on the set of lattices by saying that two lattices $L$ and $L'$ are equivalent if there is an element $\lambda \in F^*$ with $L = \lambda L'$. The equivalence class of the lattice $L$ is denoted $[L]$. The action of $GL_n(F)$ preserves this relation, since for any $g \in GL_n(F)$

$$gL = g(\lambda L') = \lambda gL',$$

and hence $GL_n(F)$ acts naturally on the set of equivalence classes by setting $g[L] = [gL]$. The stabilizer of the lattice class $[[B]]$ is easily seen to be $Z \cdot GL_n(O)_{B}$, where $Z$ is the subgroup of $GL_n(F)$ of diagonal matrices with all entries equal, i.e. of the form diag($\lambda, \ldots, \lambda$), $\lambda \in F^*$. We denote the lattice class of the standard lattice by $\Lambda_0$. 


Index of notation

\begin{itemize}
  \item $S_n$ \hspace{1cm} The symmetric group on $n$ letters
  \item $D_{2n}$ \hspace{1cm} The dihedral group of order $2n$
  \item $D_\infty$ \hspace{1cm} The infinite dihedral group
  \item $\langle S \rangle$ \hspace{1cm} The group generated by the elements of $S$
  \item $\Sigma(W, S)$ \hspace{1cm} The Coxeter complex of the Coxeter group $W$ with generating set $S$
  \item $[x, \sigma]$ \hspace{1cm} The element of $\mathbb{Z}^n \rtimes S_{n+1}$, represented by the elements $x \in \mathbb{Z}^n$ and $\sigma \in S_{n+1}$
  \item $\langle x, y \rangle$ \hspace{1cm} The subspace spanned by the vectors $x$ and $y$.
  \item $\langle p, q \rangle$ \hspace{1cm} The path from the points $p$ and $q$ of an $\mathbb{R}$-tree.
  \item $\text{GL}_n(F)$ \hspace{1cm} The general linear group of $F^n$
  \item $\text{SL}_n(F)$ \hspace{1cm} The special linear group of $F^n$
  \item $\mathcal{O}^n$ \hspace{1cm} The $\mathcal{O}$-lattice spanned by the standard basis of $F^n$
  \item $[B]$ \hspace{1cm} The $\mathcal{O}$-lattice spanned by the basis $B$
  \item $\text{GL}_n(\mathcal{O})$ \hspace{1cm} The subgroup of $\text{GL}_n(F)$ that stabilizes $\mathcal{O}^n$
  \item $\text{GL}_n(\mathcal{O})_B$ \hspace{1cm} The subgroup of $\text{GL}_n(F)$ that stabilizes the $\mathcal{O}$-lattice spanned by the basis $B$
  \item $M^*_n$ \hspace{1cm} The group of invertible matrices with entries in $\mathcal{O}$
  \item $[L]$ \hspace{1cm} The lattice class of the lattice $L$
  \item $A_0$ \hspace{1cm} The lattice class of the lattice $\mathcal{O}^n$
\end{itemize}
Bibliography


