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Tensor products on Category O and Kostant's problem

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Abstract

Kåhrström, J. 2008. Tensor Products on Category O and Kostant's Problem. *Uppsala Dissertations in Mathematics* 59. 36 pp. Uppsala. ISBN 978-91-506-2034-4.

This thesis consists of a summary and three papers, concerning some aspects of representation theory for complex finite dimensional semi-simple Lie algebras with focus on the BGG-category O.

Paper I is motivated by the many useful properties of functors on category O given by tensoring with finite dimensional modules, such as projective functors and translation functors. We study properties of functors on O given by tensoring with arbitrary (possibly infinite dimensional) modules. Such functors give rise to a faithful action of O on itself via exact functors which preserve tilting modules, via right exact functors which preserve projective modules, and via left exact functors which preserve injective modules.

Papers II and III both deal with Kostant's problem. In Paper II we establish an effective criterion equivalent to the answer to Kostant's problem for simple highest weight modules, in the case where the Lie algebra is of type A. Using this, we derive some old and new results which answer Kostant's problem in special cases. An easy sufficient condition derived from this criterion using Kazhdan-Lusztig combinatorics allows for a straightforward computational check using a computer, by which we get a complete answer for simple highest weight modules in the principal block of O for algebras of rank less than 5.

In Paper III we relate the answer to Kostant's problem for certain modules to the answer to Kostant's problem for a module over a subalgebra. We also give a new description of a certain quotient of the dominant Verma module, which allows us to give a bound on the multiplicities of simple composition factors of primitive quotients of the universal enveloping algebra.

Keywords: Semi-simple Lie algebras, Tensor products, Kostant's problem, Primitive quotients

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Black
then
white are
all i see
in my infancy.
Red and yellow then came to be,
reaching out to me.
Lets me see.

As below, so above and beyond, I imagine
drawn beyond the lines of reason.
Push the envelope.
Watch it bend.

Over thinking, over analyzing
    separates the body from the mind.
Withering my intuition,
    missing opportunities and I must
Feed my will to feel my moment
    drawing way outside the lines.

Lateralus, Tool

List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Kåhrström, J. (2008) Tensoring with infinite-dimensional modules in $\mathcal{O}_0$, to appear in *Algebras and Representation Theory.*

II Kåhrström, J., Mazorchuk, V. (2008) A new approach to Kostant's problem, Manuscript.

III Kåhrström, J. (2008) Kostant's problem and parabolic subgroups, Manuscript.

Contents

1. Introduction

The representation theory of Lie algebras and Lie groups is a thriving area of research, with applications in a wide range of areas such as functional analysis, number theory, theoretical physics and chemistry, among others. Throughout this thesis, we fix a complex finite-dimensional semi-simple Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$, with a fixed triangular decomposition. The full category of $\mathfrak{g}$-modules is very complicated in general, for example there is no classification of simple $\mathfrak{g}$-modules for $\mathfrak{g} \neq \mathfrak{sl}_2$. The theory for finite dimensional modules is well understood, but for general modules the situation is complicated.

1.1 The BGG category $\mathcal{O}$

The category $\mathcal{O}$, first defined in [BGG76], is the category of finitely generated weight $\mathfrak{g}$-modules that are locally $\mathcal{U}(\mathfrak{n})$-finite. For a comprehensive review of category $\mathcal{O}$, its history and main problems, Humphreys' recent book [Hum08] is a good source.

Category $\mathcal{O}$ is a natural extension of the category of finite dimensional $\mathfrak{g}$-modules. It is 'small enough' to be effectively studied, yet it contains valuable information on the general representation theory of $\mathfrak{g}$ and its enveloping algebra $\mathcal{U}(\mathfrak{g})$. For example, by a classical result of Duflo [Duf77], the primitive ideals of $\mathcal{U}(\mathfrak{g})$, i.e. ideals given as the annihilator $\operatorname{Ann} M$ of some simple $\mathfrak{g}$-module M, are all on the form $\operatorname{Ann} L$, where L is a simple module in $\mathcal{O}$.

To each weight $\lambda \in \mathfrak{h}^*$, there is an associated simple highest weight $\mathfrak{g}$-module $L(\lambda)$ and a Verma module $\Delta(\lambda)$, and each Verma module has a finite composition series by simple highest weight modules. In category $\mathcal{O}$ each simple module $L(\lambda)$ has in addition a projective cover $P(\lambda)$, which in turn is filtered by Verma modules. This filtration satisfies the remarkable *BGG-reciprocity* (where the left hand side denotes the multiplicity of $\Delta(\mu)$ in the Verma-filtration of $P(\lambda)$),

$$\big(P(\lambda) \colon \Delta(\mu)\big) = \big[\Delta(\mu) \colon L(\lambda)\big].$$

The BGG-reciprocity is mentioned in [BGG76] as the main motivation behind the definition of the category $\mathcal{O}$.

Determination of the composition factors of Verma modules and their multiplicities was an early fundamental problem. The category $\mathcal{O}$ gives

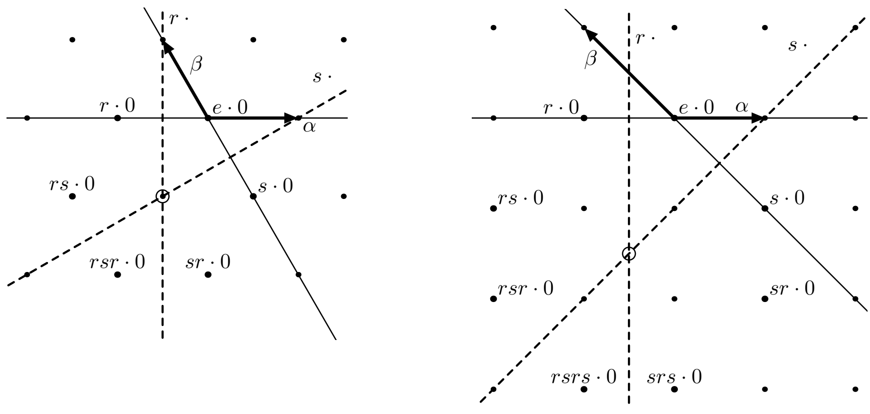

Figure 1.1: The orbit of the zero weight under the dot action of the Weyl group of $\mathfrak{sl}_3$ (left) and $\mathfrak{so}_3$ (right). The simple roots are marked as α and β, and the corresponding simple reflections by r and s, with the dashed lines being their lines of reflection. The dots show the integral span of the roots, and the circles show half the sum of the negative roots.

a good framework for studying this problem (also noted in [BGG76] as a motivation for studying the category): it decomposes into a direct sum of *blocks*, indexed by dominant weights, and each such block is equivalent to the module category over a finite dimensional quasi-hereditary associative algebra. Such categories are much better understood than the general theory of $\mathfrak{g}$-modules, which provides a substantial simplification of the problem. This approach proved successful, and culminated in the Kazhdan-Lusztig conjecture three years later [KL79] (which by now has been a well established theorem for almost thirty years, proved in [BB81] and [BK81]).

There are a lot of similarities between the different blocks, as all regular blocks corresponding to integral weights are equivalent, and the remaining blocks are in a sense degenerate versions. For this reason we will mainly restrict our attention to the 'most complicated' *principal block* $\mathcal{O}_0$, containing the trivial module, and we let A denote the corresponding finite dimensional algebra. The highest weights occurring in modules in this category are given by the orbit of the zero weight under the so-called 'dot action' of the Weyl group W, given on a weight $\lambda \in \mathfrak{h}^*$ by

$$w \cdot \lambda := w(\lambda + \rho) - \rho,$$

where ρ is half the sum of the positive roots. Figure 1.1 shows this orbit for the algebras $\mathfrak{sl}_3$ and $\mathfrak{so}_3$.

For each element $x \in W$, we get a simple module $L(x)$ and a Verma module $\Delta(x)$ both of highest weight $x{\cdot}0$, and a projective module $P(x)$ —

the projective cover of $L(x)$ (Figure 1 in Paper I illustrates the non-zero weight spaces of each of the simple modules in $\mathcal{O}_0$ for the Lie algebra $\mathfrak{sl}_3$). These three families of modules each provide a basis for the Grothendieck group of $\mathcal{O}_0$, and the Kazhdan-Lusztig conjecture provides a recursive method for finding the transformation matrix for changing between these bases.

At the heart of the Kazhdan-Lusztig conjecture lies the Iwahori-Hecke algebra $\mathcal{H}$ of W, which is the free $\mathbb{Z}[v, v^{-1}]$-module over the basis $\{ H_x \mid x \in W \}$ with the relations

$$\begin{aligned} H_s H_s &= H_e + (v^{-1} - v) H_s && \text{for simple reflections } s \text{, and} \\ H_x H_y &= H_{xy} && \text{if } \ell(xy) = \ell(x) + \ell(y). \end{aligned}$$

For each $x \in W$, Kazhdan and Lusztig construct an element

$$\underline{H}_x \in H_x + \sum_{y \in W} v\mathbb{Z}[v] H_y,$$

which is fixed under the involution mapping v to v^{-1} and mapping H_y to $H_{y^{-1}}^{-1}$ for $y \in W$. The set $\{ \underline{H}_x \mid x \in W \}$ is a basis for $\mathcal{H}$, called the Kazhdan-Lusztig basis. By expressing the elements of this basis in the standard basis,

$$\underline{H}_x = \sum_{y \in W} h_{y,x}(v) H_y,$$

we obtain a set of polynomials in the indeterminate v, called the Kazhdan-Lusztig polynomials. One form of the Kazhdan-Lusztig conjecture now states that

$$\big[\Delta(y) \colon L(x)\big] = h_{y,x}(1),$$

hence giving an answer to the question of composition factors of Verma modules.

This connection between category $\mathcal{O}_0$ and the Hecke algebra of W can be greatly refined as follows, where we implicitly use the equivalence of $\mathcal{O}_0$ with the category of A-modules. The algebra A has a natural positive $\mathbb{Z}$-grading, and we denote by $A^{\mathbb{Z}}$ the corresponding $\mathbb{Z}$-covering. We say that an A-module M has a *graded lift* if there exists an $A^{\mathbb{Z}}$-module that is isomorphic to M as an A-module after forgetting the grading, and we denote the category of graded modules by $\mathcal{O}_0^{\mathbb{Z}}$. Not all modules in $\mathcal{O}_0$ have graded lifts, but in [BGS96] it is shown that the simple modules, Verma modules and projective modules all have graded lifts. Furthermore, the Grothendieck group of $\mathcal{O}_0^{\mathbb{Z}}$ is isomorphic to the Hecke algebra $\mathcal{H}$, where a shift in grading corresponds to multiplication by v, and

$$\begin{aligned}[L(x)] &= \hat{\underline{H}}_x, \\ [\Delta(x)] &= H_x, \text{ and,} \\ [P(x)] &= \underline{H}_x.\end{aligned}$$

Here, $\hat{\underline{H}}_x$ denotes the elements of the *dual* Kazhdan-Lusztig basis, which satisfies

$$\tau\big(\underline{H}_x \hat{\underline{H}}_y\big) = \delta_{x,y^{-1}},$$

where $\tau\colon \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ is the *symmetrizing trace* defined by

$$\tau(H_x) = \delta_{x,e}.$$

Since $A^{\mathbb{Z}}$ is positively graded, the grading of a module translates to a filtration of the corresponding ungraded module by submodules (where in addition each composition factor is semi-simple). Hence the graded picture contains a large amount of extra information about the corresponding modules.

For example, in the case $\mathfrak{g} = \mathfrak{sl}_3$, we have $W = S_3$; the simple reflections are $r = (1\ 2)$ and $s = (2\ 3)$; and the elements of W are e, r, s, rs, sr, and rsr (cf. left part of Figure 1.1). The Kazhdan-Lusztig basis and dual Kazhdan-Lusztig basis of $\mathcal{H}$ can be computed fairly easily by hand:

$$\begin{aligned}\underline{H}_e &= H_e, \\ \underline{H}_r &= H_r + vH_e, \\ \underline{H}_s &= H_s + vH_e, \\ \underline{H}_{rs} &= H_{rs} + vH_r + vH_s + v^2H_e, \\ \underline{H}_{sr} &= H_{sr} + vH_r + vH_s + v^2H_e, \\ \underline{H}_{rsr} &= H_{rsr} + vH_{rs} + vH_{sr} + v^2H_r + v^2H_s + v^3H_e,\end{aligned}$$

and

$$\begin{aligned}\hat{\underline{H}}_e &= H_e - vH_s - vH_r + v^2H_{rs} + v^2H_{sr} - v^3H_{rsr}, \\ \hat{\underline{H}}_r &= H_r - vH_{rs} - vH_{sr} + v^2H_{rsr}, \\ \hat{\underline{H}}_s &= H_s - vH_{rs} - vH_{sr} + v^2H_{rsr}, \\ \hat{\underline{H}}_{rs} &= H_{rs} - vH_{rsr}, \\ \hat{\underline{H}}_{sr} &= H_{sr} - vH_{rsr}, \\ \hat{\underline{H}}_{rsr} &= H_{rsr}.\end{aligned}$$

Writing, for instance, H_e in the dual Kazhdan-Lusztig basis, we get

$$H_e = \hat{\underline{H}}_e + v\hat{\underline{H}}_r + v\hat{\underline{H}}_s + v^2\hat{\underline{H}}_{rs} + v^2\hat{\underline{H}}_{sr} + v^3\hat{\underline{H}}_{rsr}.$$

By interpreting this in $\mathcal{O}_0$, we see that the dominant Verma module $\Delta(e)$ has a filtration where the factors are

$$\begin{gathered} L(e), \\ L(r) \oplus L(s), \\ L(rs) \oplus L(sr), \text{ and} \\ L(rsr). \end{gathered}$$

For a more complicated example, in the case $\mathfrak{g} = \mathfrak{sl}_4$ we have $W = S_4$ and the simple reflections are $r = (1\ 2)$, $s = (2\ 3)$ and $t = (3\ 4)$. Writing the element H_e in the dual Kazhdan-Lusztig basis we get

$$\begin{aligned} H_e = {} & \underline{\hat{H}}_e + \\ & v\underline{\hat{H}}_r + v\underline{\hat{H}}_s + v\underline{\hat{H}}_t + \\ & v^2\underline{\hat{H}}_{rs} + v^2\underline{\hat{H}}_{rt} + v^2\underline{\hat{H}}_{sr} + v^2\underline{\hat{H}}_{st} + v^2\underline{\hat{H}}_{ts} + v^2\underline{\hat{H}}_{srts} + \\ & v^3\underline{\hat{H}}_{rsr} + v^3\underline{\hat{H}}_{rst} + v^3\underline{\hat{H}}_{rts} + v^3\underline{\hat{H}}_{srt} + v^3\underline{\hat{H}}_{sts} + v^3\underline{\hat{H}}_{tsr} + v^3\underline{\hat{H}}_{rstsr} + \\ & v^4\underline{\hat{H}}_{rsrt} + v^4\underline{\hat{H}}_{rsts} + v^4\underline{\hat{H}}_{rtsr} + v^4\underline{\hat{H}}_{srts} + v^4\underline{\hat{H}}_{stsr} + \\ & v^5\underline{\hat{H}}_{rsrts} + v^5\underline{\hat{H}}_{rstsr} + v^5\underline{\hat{H}}_{srtsr} + \\ & v^6\underline{\hat{H}}_{rsrtsr}, \end{aligned}$$

which tells us that the dominant verma module $\Delta(e)$ has a filtration whose factors are

$$\begin{gathered} L(e), \\ L(r) \oplus L(s) \oplus L(t), \\ L(rs) \oplus L(rt) \oplus L(sr) \oplus L(st) \oplus L(ts) \oplus L(srts), \\ L(rsr) \oplus L(rst) \oplus L(rts) \oplus L(srt) \oplus L(sts) \oplus L(tsr) \oplus L(rstsr), \\ L(rsrt) \oplus L(rsts) \oplus L(rtsr) \oplus L(srts) \oplus L(stsr), \\ L(rsrts) \oplus L(rstsr) \oplus L(srtsr), \text{ and} \\ L(rsrtsr). \end{gathered} \tag{1.1}$$

1.2 Tensor products

Given two $\mathfrak{g}$-modules M and N, we can make their tensor product $M \otimes N$ into a $\mathfrak{g}$-module through

$$x(m \otimes n) := (xm) \otimes n + m \otimes (xn).$$

Fixing a finite dimensional module V, we obtain an exact functor $V \otimes \_$ on the category of $\mathfrak{g}$-modules which has many good properties. For example, the equivalence between blocks of $\mathcal{O}$ mentioned in the previous section can be proven using such functors. By composing with projection onto the principal block, we get an endofunctor on $\mathcal{O}_0$. In general these functors are decomposable, and the direct summands of such a functor is called *projective functors*. These were introduced in [BG80], where it was shown (in a more general setting) that for each $x \in W$ there exists a unique projective functor θ_x satisfying

$$\theta_x P(e) = P(x),$$

and that any projective functor on $\mathcal{O}_0$ can be written as a direct sum of such functors. This was refined in [Str03], where projective functors were shown to have graded lifts to $\mathcal{O}_0^{\mathbb{Z}}$. Furthermore, on the level of the graded Grothendieck group, θ_x acts by right multiplication with $\underline{H}_x$.

For example, if $\mathfrak{g} = \mathfrak{sl}_4$, consider the module $\theta_{rt} L(rt)$ (using the notation of the previous section). In the graded Grothendieck group we have

$$\left[\theta_{rt} L(rt)\right] = \left[L(rt)\right] \underline{H}_{rt} = \underline{\hat{H}}_{rt} \underline{H}_{rt},$$

which, written in the dual Kazhdan-Lusztig basis is

$$\begin{aligned} \underline{\hat{H}}_{rt} \underline{H}_{rt} = & \underline{\hat{H}}_e + (v^{-1} + v) \underline{\hat{H}}_r + (v^{-1} + v) \underline{\hat{H}}_t + \\ & \underline{\hat{H}}_{rs} + (v^{-2} + 2 + v^2) \underline{\hat{H}}_{rt} + \underline{\hat{H}}_{ts} + \\ & (v^{-1} + v) \underline{\hat{H}}_{rst} + (v^{-1} + v) \underline{\hat{H}}_{rts} + (v^{-1} + v) \underline{\hat{H}}_{tsr}. \end{aligned}$$

Hence the module $\theta_{rt} L(rt)$ has a filtration whose factors are

$$\begin{gathered} L(rt), \\ L(r) \oplus L(t) \oplus L(rst) \oplus L(rts) \oplus L(tsr), \\ L(e) \oplus L(rs) \oplus L(rt) \oplus L(rt) \oplus L(ts), \\ L(r) \oplus L(t) \oplus L(rst) \oplus L(rts) \oplus L(tsr), \text{ and} \\ L(rt). \end{gathered} \tag{1.2}$$

1.3 Kostant's problem

For two $\mathfrak{g}$-modules M and N, the space $\mathrm{Hom}_{\mathbb{C}}(M, N)$ of linear maps from M to N has a natural $\mathcal{U}(\mathfrak{g})$-bimodule structure given by

$$(ufv)(m) := uf(vm),$$

for $u, v \in \mathcal{U}(\mathfrak{g})$, $f \in \mathrm{Hom}_{\mathbb{C}}(M, N)$ and $m \in M$. We can thus define a $\mathfrak{g}$-module structure on $\mathrm{Hom}_{\mathbb{C}}(M, N)$ through the adjoint action, i.e.

$$x \cdot f := xf - fx,$$

for $x \in \mathfrak{g}$. The $\mathfrak{g}$-submodule of $\mathrm{Hom}_{\mathbb{C}}(M, N)$ consisting of locally finite elements is in fact a $\mathcal{U}(\mathfrak{g})$-sub-bimodule, which we denote by $\mathcal{L}(M, N)$. Such modules are interesting for a number of reasons. For example, they are Harish-Chandra bimodules; the *principal series* modules can be expressed as $\mathcal{L}\big(\Delta(\lambda), \nabla(\mu)\big)$ (where $\nabla(\mu) = \Delta(\mu)^\star$, in which $\star$ denotes the simple-preserving dual on $\mathcal{O}$); and they can be used to define the twisting and completion functors ([Jos82], [Jos83], [KM05]).

We are mainly interested in studying $\mathcal{L}(M, M)$ for a module $M \in \mathcal{O}$, in which case it becomes a Noetherian ring. Since $\mathcal{U}(\mathfrak{g})$ is locally finite under the adjoint action of $\mathfrak{g}$ it follows, for $u \in \mathcal{U}(\mathfrak{g})$, that the action of u on M gives an element of $\mathcal{L}(M, M)$. Hence we have a homomorphism of rings

$$\mathcal{U}(\mathfrak{g}) \to \mathcal{L}(M, M),$$

whose kernel is $\mathrm{Ann}\, M$. Kostant posed the question whether the induced injection

$$\mathcal{U}(\mathfrak{g}) / \mathrm{Ann}\, M \hookrightarrow \mathcal{L}(M, M)$$

is a surjection for all simple modules M.

Trivially, it has the positive answer for $L(e)$, but in general it is a difficult problem, for which the answer is not even known for simple highest weight modules. Early examples where the answer is negative were found by Conze-Berline and Duflo ([CBD77]) in cases when $\mathfrak{g}$ has roots of different length. The question can also be of interest for non-simple modules, and one of the early positive results was that it has the positive answer for all Verma modules, as well as for all quotients of the dominant Verma module $\Delta(e)$. In particular, it has the positive answer for $L(w_\circ)$, where $w_\circ \in W$ is the longest element of the Weyl group. This was generalized by Gabber and Joseph in [GJ81], where they show that Kostant's problem has the positive answer for $L(w_\circ^I w_\circ)$, where $w_\circ^I$ is the longest element of a parabolic subgroup W_I of W. In [Maz05] Mazorchuk showed that it has the positive answer for $L(s)$, where s is a simple reflection of W, as well as for $L(s w_\circ^I w_\circ)$ if $s \in W_I$. Furthermore, if $\mathfrak{g}$ is of type A, Mazorchuk and Stroppel have shown ([MS08a]) that the answer to Kostant's problem is a *left cell* invariant[1]. In [MS08b], Stroppel and Mazorchuk also showed that Kostant's problem has the *negative* answer for the $\mathfrak{sl}_4$-module $L(rt)$ (notation as in Section 1.2). This was quite surprising, since Joseph has shown that the fields of fractions for $\mathcal{U}(\mathfrak{g}) / \mathrm{Ann}\, L(x)$ and $\mathcal{L}\big(L(x), L(x)\big)$ are isomorphic for all $x \in W$ if $\mathfrak{g}$ is of type A.

[1] See Section 2.2 for the definition of left cells.

2. Summary of papers

In this section a short summary of each paper contained in the thesis is presented. The first paper concerns tensoring with infinite dimensional modules, while the second and third papers both deal with Kostant's problem.

2.1 Paper I

Motivated by the many applications of tensoring with finite dimensional modules, we investigate tensor products of (possibly infinite dimensional) modules in $\mathcal{O}$, with focus on the subcategory $\mathcal{O}_0$. For two modules M, $N \in \mathcal{O}$, their tensor product $M \otimes N$ is only an element of $\mathcal{O}$ if at least one of the modules M or N is finite dimensional. However, when both modules M and N are infinite dimensional, the module $M \otimes N$ still has well defined projections onto the blocks of $\mathcal{O}$. In particular, given a module $M \in \mathcal{O}_0$, we can define an exact endofunctor G_M on $\mathcal{O}_0$ by

$$
\begin{aligned}
G_M \colon \mathcal{O}_0 &\to \mathcal{O}_0, \\
N &\mapsto \big(M \otimes N\big)\!\downarrow_0 && \text{on modules, and} \\
\varphi &\mapsto (\mathrm{id} \otimes \varphi)\downarrow_0 && \text{on morphisms,}
\end{aligned}
$$

where $\downarrow_0$ denotes projection onto the principal block.

In general these functors do not behave as well as projective functors. For example, if M is a finite dimensional module, then M^* is also finite dimensional, and the right and left adjoint of $M \otimes \_\_$ is $M^* \otimes \_\_$, again a projective functor. This implies that such functors take projective modules to projective modules, tilting modules to tilting modules, and injective modules to injective modules. If on the other hand M is an infinite dimensional module in $\mathcal{O}_0$, the module M^* is no longer locally $\mathcal{U}(\mathfrak{n})$-finite and is hence no longer in $\mathcal{O}_0$. The right and left adjoints of G_M (which we denote by F_M and H_M, respectively) are thus essentially very different types of functors. These three classes of functors still share some of their properties with the projective functors. For instance, the functors F_M map projective modules to projective modules, the functors G_M map tilting modules to tilting modules, and the functors H_M map injective modules to injective modules.

Analogous to the similar case in [Nei89] and [Fie03], where translation functors on the category $\mathcal{O}$ for Kac-Moody algebras are studied, we have that

$$F_M N = \left(M^* \otimes N\right)^{\leqslant 0}\big\downarrow_0,$$

where $L^{\leqslant 0}$ essentially means 'the biggest quotient of L contained in $\mathcal{O}_0$'. Using the simple-preserving duality $\star$ on $\mathcal{O}$, we also find that

$$H_M N = \left(F_{M^\star} N^\star\right)^\star.$$

For each morphism $\varphi : M \to N$ between objects in $\mathcal{O}_0$ we obtain corresponding natural transformations

$$\begin{aligned} &F_\varphi\colon F_N \to F_M, \\ &G_\varphi\colon G_M \to G_N, \text{ and} \\ &H_\varphi\colon H_N \to H_M. \end{aligned}$$

Hence, F, G and H are functors from category $\mathcal{O}_0$ to the category of endofunctors on $\mathcal{O}_0$, where G is covariant and F and H are contravariant.

Denote by $\mathrm{PFun}(\mathcal{O}_0)$ the category of endofunctors on $\mathcal{O}_0$ preserving the additive subcategory of projective modules. Similarly, denote by $\mathrm{TFun}(\mathcal{O}_0)$ and $\mathrm{IFun}(\mathcal{O}_0)$ the corresponding categories preserving the additive subcategories of tilting and injective modules, respectively. We then have the following main theorem.

Theorem 2.1. *F, G and H define faithful functors*

$$\begin{aligned} F &: \mathcal{O}_0 \hookrightarrow \mathrm{PFun}(\mathcal{O}_0), M \mapsto F_M, \\ G &: \mathcal{O}_0 \hookrightarrow \mathrm{TFun}(\mathcal{O}_0), M \mapsto G_M, \\ H &: \mathcal{O}_0 \hookrightarrow \mathrm{IFun}(\mathcal{O}_0), M \mapsto H_M, \end{aligned}$$

all three satisfying $X_M \cong X_N$ if and only if $M \cong N$ (where $X = F, G, H$).

These functors can be used to 'create' projective, tilting and injective modules, as in the following proposition.

Proposition 2.2. *For all modules M, $N \in \mathcal{O}_0$ such that M has a Verma filtration and N has a dual Verma filtration, we have*
- *(a) $F_N M$ is projective,*
- *(b) $G_M N \cong G_N M$ is a tilting module, and*
- *(c) $H_M N$ is injective.*

We also show that the functor $G_{\Delta(e)}$ admits the structure of a comonad, which is a categorical equivalent of a comultiplication, and dually the functor $G_{\nabla(e)}$ admits the structure of a monad, which is a categorical equivalent of a multiplication. Furthermore, we generalize the main results of the paper to parabolic versions of category $\mathcal{O}$. Finally, we compute the 'multiplication table' of all $F_M N$ and $G_M N$, where M and N are simple modules in $\mathcal{O}_0$ and $\mathfrak{g} = \mathfrak{sl}_3$.

2.2 Paper II

Paper II is mainly motivated by [MS08b], where Mazorchuk and Stroppel showed that Kostant's problem has the negative answer for the $\mathfrak{sl}_4$-module $L(rt)$ (notation as in Section 1.1). We systematize the method used in [MS08b], to give a general criterion which allows us to verify whether Kostant's problem has the positive answer for a highest weight module over a semi-simple Lie algebra of type A.

Hence, in this paper we assume that $\mathfrak{g} = \mathfrak{sl}_n$ for some $n \in \mathbb{N}$. We rely heavily on *left* and *right cells*, which in terms of the Hecke algebra are defined as follows.

For two elements x, $y \in W$, we first define the *left pre-order* on W by $x \leqslant_L y$ if

$$\mathcal{H}\underline{H}_x \subseteq \mathcal{H}\underline{H}_y, \quad \text{or equivalently,} \quad \mathcal{H}\underline{\hat{H}}_x \supseteq \mathcal{H}\underline{\hat{H}}_y,$$

and similarly the *right pre-order* by $x \leqslant_R y$ if

$$\underline{H}_x\mathcal{H} \subseteq \underline{H}_y\mathcal{H}, \quad \text{or equivalently,} \quad \underline{\hat{H}}_x\mathcal{H} \supseteq \underline{\hat{H}}_y\mathcal{H}.$$

These pre-orders satisfy $x \leqslant_R y$ if and only if $x^{-1} \leqslant_L y^{-1}$.

The equivalence classes of these pre-orders are the left and right cells, respectively. These order relations are closely linked to $\mathcal{O}_0$, as $x \leqslant_L y$ if and only if

$$\operatorname{Ann} L(x) \supseteq \operatorname{Ann} L(y),$$

and $x \leqslant_R y$ if and only if $L(x)$ occurs as a composition factor of $\theta L(y)$ for some projective functor θ. The right order for S_3 and S_4 is shown in Figure 2.1.

In type A, any pair of left and right cells intersect in at most one element. This was used in [MS08a] (Proposition 60) to show that

$$\mathcal{L}\big(L(x), L(x)\big) \cong \mathcal{L}\big(L(y), L(y)\big)$$

if $x \simeq_L y$. In particular, the answer to Kostant's problem for $L(x)$ is a left cell invariant, since $x \simeq_L y$ implies $\operatorname{Ann} L(x) = \operatorname{Ann} L(y)$. Furthermore, in type A each left and right cell contains a unique involution, and hence it suffices to consider Kostant's problem for involutions.

Fixing an involution $d \in W$, there is a unique quotient D of $\Delta(e)$ which satisfies $\operatorname{Ann} D = \operatorname{Ann} L(d)$. It can be described as the unique non-zero homomorphic image of $\Delta(e)$ in $\theta_d L(d)$. The module D has $L(d)$ as its only simple submodule, and

$$\big[D\colon L(x)\big] \neq 0 \text{ implies } x \leqslant_R d,$$

and if $x \simeq_R d$ then

$$\big[D\colon L(x)\big] = \delta_{x,d}.$$

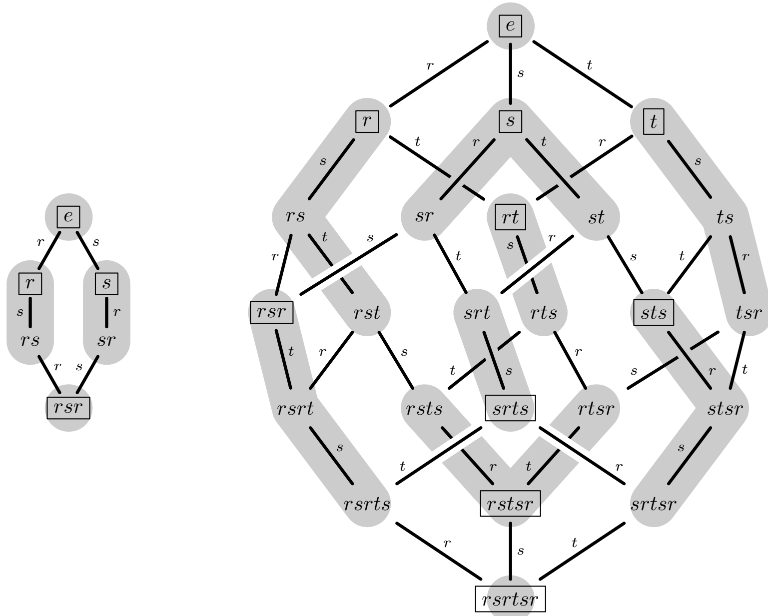

Figure 2.1: Right order of S_3 (left) and S_4 (right), where each cell is marked in grey. If cell A can be reached from cell B by strictly moving down in the diagram, then $A >_R B$. The lines give the right multiplication by simple reflections, and the involutions are boxed.

Using these properties we show that the inclusion $L(d) \hookrightarrow D$ extends to an inclusion

$$\mathcal{L}(D, D) \hookrightarrow \mathcal{L}\big(L(d), L(d)\big). \tag{2.1}$$

Since D is a quotient of $\Delta(e)$, Kostant's problem has the positive solution for D. Hence Kostant's problem has the positive solution for $L(d)$ if and only if the inclusion (2.1) is an isomorphism.

To be able to compare $\mathcal{L}\big(L(d), L(d)\big)$ and $\mathcal{L}(D, D)$ we introduce the category $\mathcal{O}_0^{\hat{\mathrm{R}}}$, which is the full subcategory of $\mathcal{O}_0$ consisting of modules $M \in \mathcal{O}_0$ such that

$$\big[M \colon L(x)\big] \neq 0 \text{ implies } x \leqslant_R d.$$

The category $\mathcal{O}_0^{\hat{\mathrm{R}}}$ contains the modules $L(d)$, D and $\theta_d L(d)$. Furthermore, by [MS08b, Key statement] the module $\theta_d L(d)$ is both the injective hull and projective cover of $L(d)$ in $\mathcal{O}_0^{\hat{\mathrm{R}}}$. We now use the *partial approximation functor* (defined in [KM05, 2.4]) with respect to a certain self-dual injective module in $\mathcal{O}_0^{\hat{\mathrm{R}}}$. Denoting this functor by A, the module $AL(d)$ can be described as the maximal extension of $L(d)$ by modules $L(x)$, where $x <_R d$.

By the properties of D and the fact that $\theta_d L(d)$ is the injective hull of $L(d)$ we obtain the list of inclusions

$$L(x) \hookrightarrow D \hookrightarrow AL(d) \hookrightarrow \theta_d L(d). \tag{2.2}$$

Furthermore, A bijectively 'lifts' elements in $\mathcal{L}\big(L(d), L(d)\big)$ up to elements in $\mathcal{L}\big(AL(d), AL(d)\big)$, i.e.

$$\mathcal{L}\big(L(d), L(d)\big) \cong \mathcal{L}\big(AL(d), AL(d)\big).$$

By (2.1) we now see that the inclusion $D \hookrightarrow AL(d)$ induces an inclusion

$$\mathcal{L}(D, D) \hookrightarrow \mathcal{L}\big(AL(d), AL(d)\big). \tag{2.3}$$

Finally, the module D is 'close enough' to being projective (i.e. up to a submodule containing only simple composition factors on the form $L(x)$ where $x <_R d$), which ensures that the inclusion (2.3) is an isomorphism precisely when $D \cong AL(d)$.

Hence, Kostant's problem has the positive answer for $L(d)$ if and only if $D \cong AL(d)$. By the definition of A, the fact that $\theta_d L(d)$ is injective, and since we have the string of inclusions (2.2), we obtain the main theorem of Paper II (slightly rephrased here):

Theorem 2.3. *Kostant's problem has the positive answer for $L(d)$ if and only if every simple submodule of the cokernel of the canonical non-zero map*

$$\Delta(e) \to \theta_d L(d)$$

is on the form $L(x)$, where $x \simeq_R d$.

Applying this criterion, we prove the (known) results that Kostant's problem has the positive answer for the socle of dominant parabolic Verma modules, as well as for $L(s)$ where s is a simple reflection[1]. Furthermore, we also prove the new result that if r and s are two commuting simple reflections, then Kostant's problem has the positive answer for the module $L(rs)$ if and only if the vertices of the Dynkin diagram representing r and s are separated by at least a distance of 3.

This criterion can also, in Lie algebras of small rank, be checked by working in the graded Grothendieck group. For example, consider the $\mathfrak{sl}_4$-module $L(rt)$[2] (using the notation in Section 1.1). The only element other than rt in the same right cell as rt is rts, and the elements strictly smaller in the right pre-order are e, r, t, rs, ts, rst and tsr

[1]These results hold in any type, whereas our proofs only work in type A.

[2]Kostant's problem has the negative answer for $L(rt)$ by the previous paragraph, but we show this here by direct computation using Theorem 2.3 for illustrative purposes.

(see Figure 2.1). Comparing the filtration (1.1) of $\Delta(e)$ with the filtration (1.2) of $\theta_{rt}L(rt)$ we conclude that the image D of the non-zero map $\Delta(e) \to \theta_{rt}L(rt)$ has a filtration

$$\begin{gathered} L(e), \\ L(r) \oplus L(t), \text{ and} \\ L(rt), \end{gathered}$$

and thus the cokernel has a filtration

$$\begin{gathered} L(rt), \\ L(r) \oplus L(t) \oplus L(rst) \oplus L(rts) \oplus L(tsr), \\ L(rs) \oplus L(rt) \oplus L(rt) \oplus L(ts), \text{ and} \\ L(rst) \oplus L(rts) \oplus L(tsr). \end{gathered}$$

In particular, the simple modules $L(rst)$ and $L(tsr)$ are submodules of the cokernel, and hence Kostant's problem has the negative answer for $L(rt)$. Using this kind of computations, Paper II concludes with a complete account for the answer to Kostant's problem for simple modules in $\mathcal{O}_0$ for $\mathfrak{sl}_n$, for all $n \leq 5$, and gives partial results for $\mathfrak{sl}_6$.

2.3 Paper III

In this paper we relate the answer to Kostant's problem for some of the simple modules in $\mathcal{O}_0$ over a semi-simple Lie algebra $\mathfrak{g}$ to the answer for simple modules in $\mathcal{O}_0$ over a semi-simple subalgebra $\mathfrak{g}_I$ of $\mathfrak{g}$, where I is a subset of the simple reflections in W.

The set I generates a subgroup W_I of W, called a *parabolic* subgroup. The subset of roots of $\mathfrak{g}$ spanned by the roots corresponding to I is a new root system, and $\mathfrak{g}_I$ is the corresponding semi-simple Lie algebra. For example, consider the algebra $\mathfrak{sl}_3$ with simple roots α and β, with basis

$$\underbrace{X_{-\alpha}, X_{-\beta}, X_{-\alpha-\beta},}_{\text{spans } \mathfrak{n}_-} \quad \underbrace{H_\alpha, H_\beta,}_{\text{spans } \mathfrak{h}} \quad \underbrace{X_\alpha, X_\beta, X_{\alpha+\beta},}_{\text{spans } \mathfrak{n}}$$

and with Weyl group $W = S_3$. The Weyl W group has simple reflections r and s, corresponding to the reflections mapping $\alpha \mapsto -\alpha$ and $\beta \mapsto -\beta$ respectively. Setting $I = \{r\}$, the root system determined by I is $\{\alpha, -\alpha\}$ (see Figure 2.2), and the subalgebra $\mathfrak{g}_I$ is the subalgebra with basis

$$X_{-\alpha}, H_\alpha, X_\alpha.$$

For a slightly more complicated example, consider $\mathfrak{g} = \mathfrak{so}_4$ with simple roots α, β and γ (with $|\alpha| = |\beta| > |\gamma|$) and corresponding simple reflections r, s and t. Setting $I = \{s, t\}$, the root system determined by I is of type B_2 (see Figure 2.3), and the subalgebra $\mathfrak{g}_I$ is isomorphic to $\mathfrak{so}_3$.

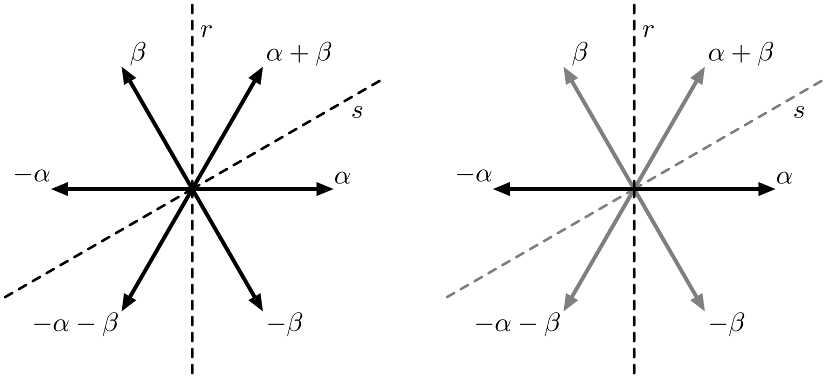

Figure 2.2: In the root system for $\mathfrak{sl}_3$, the simple reflection r determines a root system of type A_1 corresponding to a subalgebra isomorphic to $\mathfrak{sl}_2$.

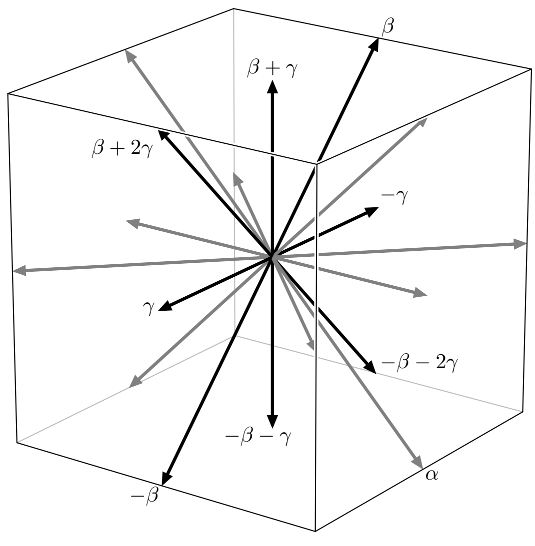

Figure 2.3: In the root system for $\mathfrak{so}_4$, the simple roots β and γ span a root system of type B_2 (the black arrows) corresponding to a subalgebra isomorphic to $\mathfrak{so}_3$.

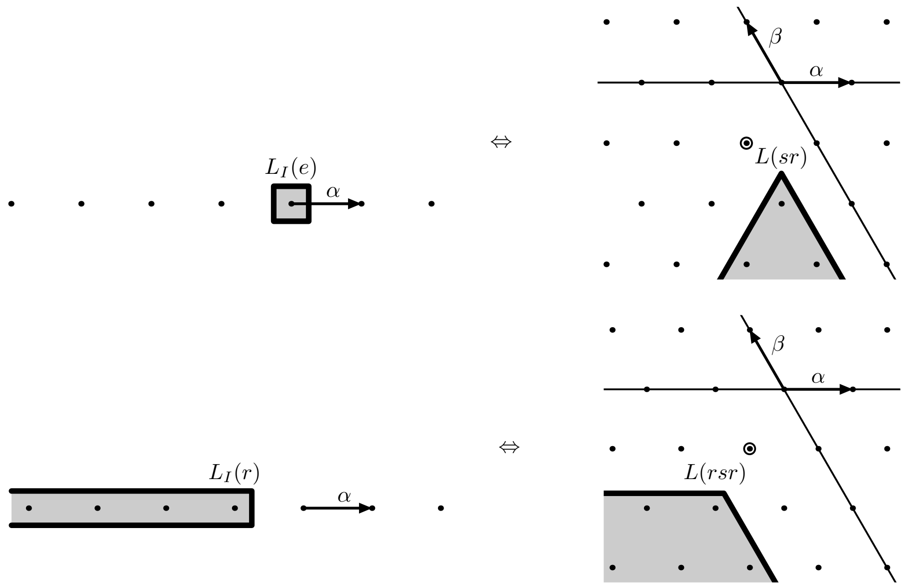

Figure 2.4: Under the equivalence between $\mathcal{O}_0^I$ and $\mathcal{O}_0^\xi$ for $\mathfrak{g} = \mathfrak{sl}_3$, where $I = \{r\}$, the module $L_I(e)$ corresponds to $L(sr)$ and the module $L_I(r)$ corresponds to $L(rsr)$. The dots mark the integral span of the roots and the grey areas mark the support of the corresponding module.

Let $\mathcal{O}_0^I$ denote the 'category $\mathcal{O}_0$' for $\mathfrak{g}_I$. We denote the simple $\mathfrak{g}_I$-module in $\mathcal{O}_0^I$ corresponding to $x \in W_I$ by $L_I(x)$, to discern it from the $\mathfrak{g}$-module $L(x)$. Let $\mathfrak{z}_I$ denote the complement of $\mathfrak{h}_I$ in $\mathfrak{h}$ with respect to the Killing form. Given $\lambda \in \mathfrak{z}_I^*$, there is a standard way to generate a module $\operatorname{Ind}_\lambda M \in \mathcal{O}$ from a module $M \in \mathcal{O}^I$, called *parabolic induction.* Conversely, from any module $M \in \mathcal{O}$ we get a module $\operatorname{Res}_\lambda M \in \mathcal{O}^I$ via restriction to the $\mathfrak{z}_I$-weight λ. Let ξ be the restriction of $w \cdot 0$ to $\mathfrak{z}_I$, and denote by $\mathcal{O}_0^\xi$ the full subcategory of $\mathcal{O}_0$ consisting of modules M such that

$$\big[M \colon L(x)\big] \neq 0 \text{ implies } w_\circ^I w_\circ \leq x \leq w_\circ,$$

where $\leq$ denotes the Bruhat order on W. In [Maz05], Mazorchuk showed that the functors Res_ξ and Ind_ξ define an equivalence between $\mathcal{O}_0^I$ and $\mathcal{O}_0^\xi$. The simple module $L_I(x)$ is mapped to the simple module $L(xw_\circ^I w_\circ)$ under this equivalence, illustrated in Figure 2.4 for the case $\mathfrak{g} = \mathfrak{sl}_3$ and $I = \{r\}$. Figure 2.5 shows how the elements of W_I embeds into W under the map $x \mapsto xw_\circ^I w_\circ$, where W is the Weyl group for $\mathfrak{so}_4$ and $I = \{s, t\}$.

This result motivated the main theorem of this paper:

Theorem 2.4. *Let $x \in W_I$. Then Kostant's problem has the positive answer for $L_I(x)$ if and only if Kostant's problem has the positive answer for $L(xw_\circ^I w_\circ)$.*

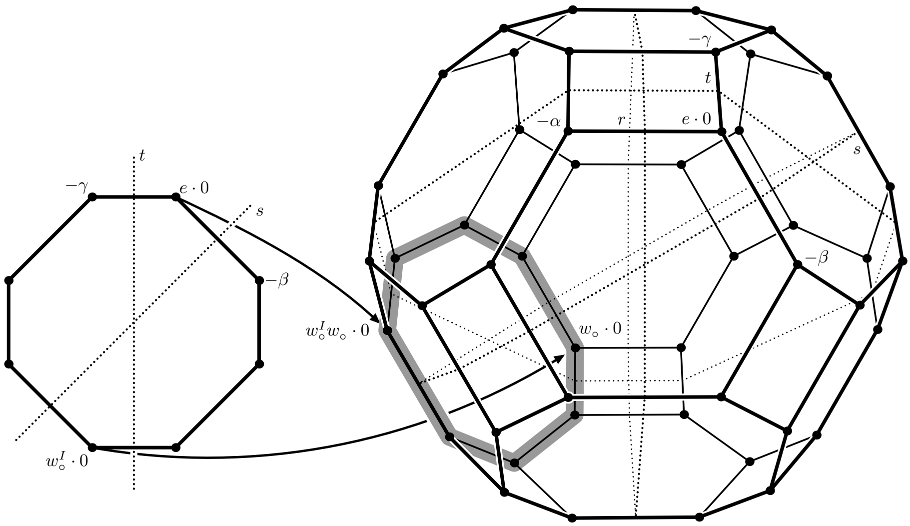

Figure 2.5: How the orbit of the zero weight for the parabolic subgroup W_I is mapped into the orbit of the zero weight for W, in the case $\mathfrak{g} = \mathfrak{so}_4$ and $I = \{s, t\}$ (compare with Figures 2.3 and 2.4).

To prove this theorem, we need to be able to compare $\mathcal{L}_I(M, M)$ and $\mathcal{L}(\operatorname{Ind}_\xi M, \operatorname{Ind}_\xi M)$ for modules $M \in \mathcal{O}_0^I$ (where again the subscript I is used to indicate that the objects are $\mathfrak{g}_I$-modules). This is accomplished via the following result (Corollary 4.2).

Proposition 2.5. *For $M, N \in \mathcal{O}_0^I$ we have*

$$\operatorname{Hom}_{\mathfrak{g}_I}\big(V, \mathcal{L}_I(M, M)\big) \cong \operatorname{Hom}_{\mathfrak{g}_I}\big(V, \mathcal{L}_I(N, N)\big)$$

for all finite dimensional $\mathfrak{g}_I$-modules V if and only if

$$\operatorname{Hom}_{\mathfrak{g}}\big(V', \mathcal{L}(\operatorname{Ind}_\xi M, \operatorname{Ind}_\xi M)\big) \cong \operatorname{Hom}_{\mathfrak{g}}\big(V', \mathcal{L}(\operatorname{Ind}_\xi N, \operatorname{Ind}_\xi N)\big)$$

for all finite dimensional $\mathfrak{g}$-modules V'.

Now fix an element $x \in W_I$. As in Paper II, there is a unique quotient D of $\Delta_I(x)$ such that $\operatorname{Ann} L_I(x) = \operatorname{Ann} D$. Since D is a quotient of the dominant Verma module, Kostant's problem has the positive answer for D, i.e.

$$\mathcal{L}_I(D, D) \cong \mathcal{U}(\mathfrak{g}_I)/\operatorname{Ann} D.$$

Since $\operatorname{Ann} D = \operatorname{Ann} L_I(x)$, we obtain an injection

$$\mathcal{L}_I(D, D) \hookrightarrow \mathcal{L}_I\big(L_I(x), L_I(x)\big), \tag{2.4}$$

which is a surjection if and only if Kostant's problem has the positive answer for $L_I(x)$. The equivalence between $\mathcal{O}_0^I$ and $\mathcal{O}_0^\xi$ gives us an injection

$$\mathcal{L}\big(\operatorname{Ind}_\xi D, \operatorname{Ind}_\xi D\big) \hookrightarrow \mathcal{L}\big(L(xw_\circ^I w_\circ), L(xw_\circ^I w_\circ)\big), \tag{2.5}$$

where $\operatorname{Ind}_\xi D$ is a quotient of the Verma module $\Delta(w_\circ^I w_\circ)$. We now use a recent (unpublished) result by Mazorchuk, that Kostant's problem has the positive answer for each quotient of $\Delta(w_\circ^I w_\circ)$, to conclude that Kostant's problem has the positive answer for $\operatorname{Ind}_\xi D$. Furthermore, the induction functor is 'nice' with respect to annihilators, i.e.

$$\operatorname{Ann}\big(\operatorname{Ind}_\xi D\big) = \operatorname{Ann}\big(\operatorname{Ind}_\xi L(x)\big),$$

and hence Kostant's problem has the positive answer for $L(xw_\circ^I w_\circ)$ if and only if the inclusion (2.5) is a surjection. Using Proposition 2.5 we can now show that the map (2.4) is a surjection if and only if the map (2.5) is a surjection, which establishes our main theorem.

Paper III concludes with some remarks on the module D used above. Under an equivalence between $\mathcal{O}_0$ and a certain category of Harish-Chandra bimodules, D corresponds to the $\mathcal{U}(\mathfrak{g})$-bimodule $\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(x)$, called a *primitive quotient* of $\mathcal{U}(\mathfrak{g})$. An important problem is the determination of simple composition factors in primitive quotients, which can thus be solved by determining the multiplicities of simple composition factors of D. Using Kazhdan-Lusztig combinatorics, we show the (aforementioned) fact that D is precisely the image of the non-zero map

$$\Delta(e) \to \theta_x L(x^{-1}).$$

Since all the above objects have graded lifts, this gives us the following bound on the multiplicity $\big[D\colon L(y)\big]$:

$$\big[D\colon L(y)\big] \leq \sum_{i\in\mathbb{Z}} \max\Big\{\big[\Delta(e)\colon L(y)\langle i\rangle\big], \big[\theta_x L(x^{-1})\colon L(y)\langle i\rangle\big]\Big\}, \tag{2.6}$$

where $L(y)\langle i\rangle$ is the standard graded lift of $L(y)$ with the grading shifted by i.

The above description of D bears striking resemblance with the description given by Duflo [Duf77, Proposition 10], where D is described as the image of a non-zero map

$$\Delta(e) \to C_{x^{-1}} \nabla(x), \tag{2.7}$$

where $C_{x^{-1}}$ is a *completion functor* (see Paper II). In [Str04] it is shown that the right hand side of (2.7) has a graded lift, which gives the following bound on the multiplicity $\big[D\colon L(y)\big]$:

$$\big[D\colon L(y)\big] \leq \sum_{i\in\mathbb{Z}} \max\Big\{\big[\Delta(e)\colon L(y)\langle i\rangle\big], \big[C_{x^{-1}}\nabla(x)\colon L(y)\langle i\rangle\big]\Big\}. \tag{2.8}$$

For algebras $\mathfrak{g}$ of small rank, the bound (2.6) is stronger than the bound (2.8), but we have not been able to prove this in general. However, Stroppel refines the bound (2.8) by using a graded version of the Duflo-Zhelobenko four step exact sequence

$$0 \to C_{x^{-1}}\nabla(sx) \to C_{(sx)^{-1}}\nabla(sx) \xrightarrow{f_{sx,x}} C_{x^{-1}}\nabla(x) \to C_{(sx)^{-1}}\nabla(x) \to 0,$$

and a similar refinement does not seem to be possible for our bound.

Summary in Swedish

Tensorprodukter på kategori $\mathcal{O}$ och Kostants problem

Denna avhandling behandlar två aspekter av representationsteorin för ändligtdimensionella halvenkla Lie-algebror; dels en studie av tensorprodukter av moduler (artikel I), dels något som kallas *Kostants problem* (artikel II och III).

När man studerar den allmänna representationsteorin för halvenkla Lie-algebror stöter man på ett problem: teorin för ändligtdimensionella moduler är relativt enkel, medan teorin för oändligtdimensionella moduler är väldigt komplicerad. I stället för att titta på allmänna oändligtdimensionella moduler begränsar vi oss därför till en kategori av moduler som kallas *kategori* $\mathcal{O}$. Den är tillräckligt 'stor' för att innehålla många intressanta oändligtdimensionella moduler, men tillräckligt 'liten' för att kunna studeras på ett effektivt sätt.

Kategori $\mathcal{O}$ har många bra egenskaper, bland annat delas den upp i olika *block*, där varje block är ekvivalent med modulkategorin för en ändligtdimensionell kvasi-ärftlig associativ algebra. Sådana kategorier är i allmänhet väl studerade, vilket innebär en avsevärd förenkling av situationen. De olika blocken är också i hög grad lika, det finns till exempel bara ett ändligt antal ekvivalensklasser under kategoriekvivalens. Av denna anledning räcker det ofta att begränsa sig till det 'mest komplicerade' blocket $\mathcal{O}_0$, blocket som innehåller den triviala modulen.

Om vi fixerar en ändligtdimensionell modul V får vi en exakt endofunktor på $\mathcal{O}$ genom

$$\begin{aligned} V \otimes \_ \colon \mathcal{O} &\to \mathcal{O}, \\ M &\mapsto V \otimes M. \end{aligned}$$

I allmänhet är en sådan funktor inte odelbar, utan den kan delas upp som en direkt summa av funktorer, och en sådan direkt summand kallas för en *projektiv* funktor. Projektiva funktorer har många bra egenskaper, bland annat kan de användas för att bevisa ekvivalensen mellan de olika blocken som nämndes ovan.

Om vi å andra sidan tar två oändligtdimensionella moduler M, $N \in \mathcal{O}$ gäller det inte längre att $M \otimes N$ tillhör $\mathcal{O}$, produkten blir för 'stor'. Däremot har produkten en väldefinierad projektion på blocket $\mathcal{O}_0$. På detta sätt får vi, givet en modul $M \in \mathcal{O}_0$, en exakt endofunktor på $\mathcal{O}_0$ definierad genom

$$\begin{aligned} G_M \colon \mathcal{O}_0 &\to \mathcal{O}_0, \\ N &\mapsto \big(M \otimes N\big)\big\downarrow_0, \end{aligned}$$

där $\downarrow_0$ betecknar projektion på block $\mathcal{O}_0$. Den första artikeln studerar dessa funktorers egenskaper. De har vissa egenskaper gemensamt med de projektiva funktorerna, men i många avseenden är de mer svårhanterliga. Till exempel har projektiva funktorer egenskapen att dess höger- och vänsteradjunkt sammanfaller, och denna funktor är igen en projektiv funktor. Vänsteradjunkten F_M och högeradjunkten H_M till funktorn G_M sammanfaller däremot inte, och de är inte direkta summander av tensorprodukter. Huvudresultatet i första artikeln är att funktorerna F, G och H var och en definierar en trogen verkan av kategori $\mathcal{O}_0$ på sig själv, och att de bevarar de additiva delkategorierna bestående av projektiva, vipp-, respektive injektiva moduler.

Den andra och tredje artikeln behandlar *Kostants problem.* Bakgrunden är följande: Låt $\mathfrak{g}$ vara den halvenkla Lie-algebra vi arbetar med, och $\mathcal{U}(\mathfrak{g})$ dess universella omslutande algebra. Givet en $\mathfrak{g}$-modul M, låt $\operatorname{Ann} M$ vara M:s annihilator, det vill säga alla element $u \in \mathcal{U}(\mathfrak{g})$ så att $uM = 0$. Definiera nu en $\mathcal{U}(\mathfrak{g})$-bimodulstruktur på vektorrumet $\operatorname{Hom}_{\mathbb{C}}(M, M)$ av linjära avbildningar på M genom

$$(ufv)(m) := uf(vm)$$

för alla u, $v \in \mathcal{U}(\mathfrak{g})$, $f \in \operatorname{Hom}_{\mathbb{C}}(M, M)$ och $m \in M$. Vidare kan vi låta $\mathfrak{g}$ verka på $\operatorname{Hom}_{\mathbb{C}}(M, M)$ genom den *adjungerade* verkan:

$$x \cdot f = xf - fx.$$

Delmängden av $\operatorname{Hom}_{\mathbb{C}}(M, M)$ vars element genererar ett ändligtdimensionellt delrum under $\mathfrak{g}$:s adjungerade verkan kan visas vara en delbimodul, som vi betecknar $\mathcal{L}(M, M)$. Givet ett element $u \in \mathcal{U}(\mathfrak{g})$ definierar u:s verkan på M precis ett sådant element, så vi får en ringhomomorfism från $\mathcal{U}(\mathfrak{g})$ till $\mathcal{L}(M, M)$, vars kärna är $\operatorname{Ann} M$. Alltså har vi en inklusion

$$\mathcal{U}(\mathfrak{g}) / \operatorname{Ann} M \hookrightarrow \mathcal{L}(M, M).$$

Bertram Kostant ställde frågan: för vilka enkla $\mathfrak{g}$-moduler är ovanstående inklusion är en bijektion? Detta har visat sig vara en mycket svår fråga som hittills inte har fått något allmänt svar, inte ens för högsta viktmoduler.

Artikel II begränsar sig till fallet då $\mathfrak{g} = \mathfrak{sl}_n$ för något $n \in \mathbb{N}$. Vi visar där ett allmänt kriterium för svaret till Kostants problem för en enkel modul i $\mathcal{O}_0$ i termer av kokärnan till en speciell kanonisk avbildning (avbildningen (*) nedan). Med hjälp av detta kriterium bevisar vi några gamla och nya resultat. För Lie-algebror av liten rang kan detta kriterium studeras med hjälp av Kazhdan-Lusztig-kombinatorik, och artikeln avslutas med en full analys av svaret till Kostants problem för alla enkla moduler i $\mathcal{O}_0$ för $\mathfrak{sl}_2$, $\mathfrak{sl}_3$, $\mathfrak{sl}_4$ och $\mathfrak{sl}_5$, samt partiella resultat för $\mathfrak{sl}_6$.

I artikel III studerar vi hur svaret på Kostants problem för vissa enkla $\mathfrak{g}$-moduler relaterar till svaret för enkla $\mathfrak{g}_I$-moduler, där $\mathfrak{g}_I$ är en speciell sorts delalgebra till $\mathfrak{g}$, definierad som följer. Låt W vara $\mathfrak{g}$:s Weylgrupp, och låt S vara W:s enkla reflektioner. En delmängd $I \subseteq S$ genererar en *parabolisk* delgrupp W_I till W, och de rötter som hör till I ger oss delalgebran $\mathfrak{g}_I$, vars Weylgrupp är precis W_I.

De enkla $\mathfrak{g}$-modulerna i $\mathcal{O}_0$ indexeras av elementen in W, och vi låter $L(x)$ vara den modul som hör till $x \in W$. Detsamma gäller för $\mathfrak{g}_I$-modulerna i motsvarande kategori, och vi låter $L_I(x)$ vara den $\mathfrak{g}_I$-modul som hör till $x \in W_I$. Huvudresultatet i artikel III säger att svaret till Kostants problem för $\mathfrak{g}_I$-modulen $L_I(x)$ är detsamma som svaret till Kostants problem för $\mathfrak{g}$-modulen $L(xw_\circ^I w_\circ)$, där $w_\circ^I$ är det längsta elementet i W_I och $w_\circ$ är det längsta elementet i W.

Artikel III avslutas med några resultat angående en modul D som används för att bevisa ovanstående sats. Givet $x \in W$ är modulen D den unika kvoten av den dominanta Vermamodulen $\Delta(e)$ i $\mathcal{O}_0$ som uppfyller $\operatorname{Ann} D = \operatorname{Ann} L(x)$. Vi visar att D är precis bilden av avbildningen

$$\Delta(e) \to \theta_x L(x^{-1}). \qquad (*)$$

Detta är intressant eftersom modulen D är, under en viss kategoriekvivalens, ekvivalent med den *primitiva kvoten* $\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(x)$. Med hjälp av Kazhdan-Lusztig-kombinatorik ger detta oss en övre begränsning för de enkla sammansättningsfaktorerna i D.

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Bibliography

[BB81] A. Beĭlinson and J. Bernstein. Localisation de $\mathfrak{g}$-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, **292** (1), pp. 15–18, 1981.

[BG80] J. N. Bernstein and S. I. Gel′fand. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. *Compositio Math.*, **41** (2), pp. 245–285, 1980.

[BGG76] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand. A certain category of $\mathfrak{g}$-modules. *Funkcional. Anal. i Priložen.*, **10** (2), pp. 1–8, 1976.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, **9** (2), pp. 473–527, 1996.

[BK81] J.-L. Brylinski and M. Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Invent. Math.*, **64** (3), pp. 387–410, 1981.

[CBD77] N. Conze-Berline and M. Duflo. Sur les représentations induites des groupes semi-simples complexes. *Compositio Math.*, **34** (3), pp. 307–336, 1977.

[Duf77] M. Duflo. Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple. *Ann. of Math. (2)*, **105** (1), pp. 107–120, 1977.

[Fie03] P. Fiebig. Centers and translation functors for the category $\mathcal{O}$ over Kac-Moody algebras. *Math. Z.*, **243** (4), pp. 689–717, 2003.

[GJ81] O. Gabber and A. Joseph. On the Bernšteĭn-Gel′fand-Gel′fand resolution and the Duflo sum formula. *Compositio Math.*, **43** (1), pp. 107–131, 1981.

[Hum08] J. E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2008.

[Jos82] A. Joseph. The Enright functor on the Bernstein-Gel′fand-Gel′fand category $\mathcal{O}$. *Invent. Math.*, **67** (3), pp. 423–445, 1982.

[Jos83] A. Joseph. Completion functors in the $\mathcal{O}$ category. In *Noncommutative harmonic analysis and Lie groups (Marseille, 1982)*, volume 1020 of *Lecture Notes in Math.*, pp. 80–106. Springer, Berlin, 1983.

[KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, **53** (2), pp. 165–184, 1979.

[KM05] O. Khomenko and V. Mazorchuk. On Arkhipov's and Enright's functors. *Math. Z.*, **249** (2), pp. 357–386, 2005.

[Maz05] V. Mazorchuk. A twisted approach to Kostant's problem. *Glasg. Math. J.*, **47** (3), pp. 549–561, 2005.

[MS08a] V. Mazorchuk and C. Stroppel. Categorification of (induced) cell modules and the rough structure of generalized verma modules. *Adv. Math.*, **219** (4), pp. 1363–1426, 2008.

[MS08b] V. Mazorchuk and C. Stroppel. Categorification of Wedderburn's basis for $\mathbb{C}[S_n]$. *Arch. Math. (Basel)*, **91** (1), pp. 1–11, 2008.

[Nei89] W. Neidhardt. Translation to and fro over Kac-Moody algebras. *Pacific J. Math.*, **139** (1), pp. 107–153, 1989.

[Str03] C. Stroppel. Category $\mathcal{O}$: gradings and translation functors. *J. Algebra*, **268** (1), pp. 301–326, 2003.

[Str04] C. Stroppel. Composition factors of quotients of the universal enveloping algebra by primitive ideals. *J. London Math. Soc. (2)*, **70** (3), pp. 643–658, 2004.